Fuchsian Groups: Intro

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1 Topological Groups

In various ways, mathematicians associate extra structure with sets and then investigate these objects. For example, by adding a binary operation to a set and validating that the set with the binary operation fulfills certain axioms, we arrive at a group. Similarly, given a set, one can specify a collection of subsets fulfilling some criteria that gives a topology, a sense of which points are close to which other points, to that set. It is reasonable that one could associate two kinds of structure with a set. This is how a topological group is defined.

Loosely, a topological group is a set which is both a group and has a topology. Certain axioms apply to ensure the group structure and topology "play nicely". The multiplication operation has to be continuous and taking inverses also has to be continuous. Since we aren't studying topological groups in particular, just rest assured that the groups we present will satisfy these axioms.

Examples
1) $(\mathbb{R}, +)$, the real numbers under addition
2) $(\mathbb{R}*, \cdot)$, the positive reals under multiplication
3) $(S^1, \cdot_{\mathbb{C}})$, the unit circle in $\mathbb{C}$ under complex multiplication
4) PSL(2, $\mathbb{R}$)

It's not immediately obvious that PSL(2, $\mathbb{R}$) is a topological group. We know it's a group; where does it's topology come from? We can take the matrix with entries a, b, c, d to the point $(a, b, c, d) \in \mathbb{R}^4$. We can give SL(2, $\mathbb{R}$) the subspace topology from $\mathbb{R}^4$. PSL(2, $\mathbb{R}$) is then just the quotient topology on SL(2, $\mathbb{R}$) with the identification $\delta : (a, b, c, d) \to (-a, -b, -c, -d)$.

We would like a simpler form of this kind of object to study first. In algebra, we study subgroups or subrings, etc. Here, we will take an object which is simpler both in the topological and in the algebraic structure of a topological group.

A *discrete subgroup* of a topological group G is a subgroup of G which, as a topological space, inherits the discrete topology from G.

Examples
1) The trivial subgroup, being a one-point set, is always a discrete subgroup.
2) Any discrete subgroup of $(\mathbb{R}, +)$ is infinite cyclic.
3) Any discrete subgroup of $(S^1, \cdot_{\mathbb{C}})$ is finite cyclic.
4) The group of transformations $T : \mathbb{H} \to \mathbb{H}$ such that $T : z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, is a discrete subgroup of PSL(2, $\mathbb{R}$).

Since any discrete subgroup of $(\mathbb{R}, +)$ is infinite cyclic and we have the isomorphism/homeomorphism $x \mapsto e^x$ into $(\mathbb{R}*, \cdot)$, any discrete subgroup of $(\mathbb{R}*, \cdot)$ is also infinite cyclic.

2 Fuchsian groups

In this section we will provide a few equivalent characterizations of the term "Fuchsian group".

A *Fuchsian group* is a discrete subgroup of PSL(2, $\mathbb{R}$).

A *Fuchsian group* is a group that acts *properly discontinuously* on the upper half plane.

I have to postpone the justification of defining the same term in two different ways while we examine the definitions of a group acting properly discontinuously on a set.

2.1 Properly discontinuous actions

Given a metric space X, a group G which is a subgroup of the isometries of X acts properly discontinuously on X if and only if any of the following three conditions hold:

1) The G orbit of any point is locally finite.

2) The G orbit of any point is discrete and the stabilizer of that point is finite.

3) For any point, there is a neighborhood of that point, V, for which only finitely many $T \in G$ satisfy $T(V) \cap V \neq \emptyset$.

To review, by the G orbit of a point $x \in X$, we mean the set $\{Tx \in X : T \in G\}$. By the stabilizer of a point $x \in X$, we mean the set $\{T \in G : Tx = x\}$. Finally, a collection of sets S_α is considered locally finite iff for any compact set K, $S_\alpha \cap K = \emptyset$ for only finitely many α.

The proof of the equivalence of these three definitions was covered in the lecture and usually involves a straightforward application of the definitions above.

2.2 Equivalence of notions of Fuchsian groups

To prove that subgroup Γ of PSL(2, $\mathbb{R}$) is discrete if and only if it acts properly discontinuously on the upper half plane, we need two lemmas.

Lemma 1: For any $x \in \mathbb{H}$ and any compact set $K \subset \mathbb{H}$, the set $\{T \in$ PSL(2, $\mathbb{R}) : Tx \in K\}$ is compact in PSL(2, $\mathbb{R}$).

Lemma 2: Let Γ be a subgroup of PSL(2, $\mathbb{R}$) action properly discontinuously on $\mathbb{H}$. Let $x \in \mathbb{H}$ be fixed by $T \in \Gamma$. Then there is a neighborhood of x, W, such that no other point of W is fixed by any transformation from Γ other than the identity.

The proofs of these lemmas can be found in the text and are hence omitted here.

Theorem A subgroup Γ of PSL(2, $\mathbb{R}$) is discrete if and only if it acts properly discontinuously on the upper half plane.

Proof ($\Rightarrow$) Let $x \in \mathbb{H}$ and $K \subset \mathbb{H}$ be compact. Then $\{T \in \Gamma : Tx \in K\} = \{T \in \mathrm{PSL}(2, \mathbb{R}) : Tx \in K\} \cap \Gamma$. The first set is compact, by lemma 1; the second is discrete, by our assumption. Therefore, the intersection of the two sets has to be finite. Therefore, we have shown the Γ orbit of any x has to be locally finite, definition (1) of properly discontinuous.

($\Leftarrow$) Let $x \in \mathbb{H}$ be such that $Tx \neq x$ for any T which is not the identity element. (Such x must exist by lemma 2.) Now, suppose for contradiction that Γ is not discrete. Then there is a sequence of T_n which converge to the identity transformation. Therefore, $T_n x \to x$, but never equal x (otherwise, this contradicts our choice of x). Hence, any disc around x has infinitely many points from the orbit of Γx. This contradicts the proper discontinuity of Γ. ■

3 Classification of elements from PSL(2, $\mathbb{R}$)

To understand PSL(2, $\mathbb{R}$) better, it helps to understand what kind of elements can be found in this group. One natural way of classifying these transformations is by their fixed-point sets. So, if we have a transformation $T : z \mapsto \frac{az+b}{cz+d}$, and we want to find the fixed points of T, we need $z = \frac{az+b}{cz+d} \Leftrightarrow cz^2 + (d-a)z - b = 0$. If $c \neq 0$, we can use the quadratic formula to solve this equation:

$$z = \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

This doesn't tell us much, so we would like to understand the discriminant better in order to see what kinds of points T can fix.

$$\begin{aligned} (d-a)^2 + 4bc &= d^2 + a^2 - 2ad + 4bc \\ &= d^2 + a^2 - 2ad + 4(ad-1) \\ &= d^2 + a^2 + 2ad - 4 \\ &= (d+a)^2 - 4 \end{aligned}$$

Hence, the type of fixed points depends only on $(d+a)^2$. We define the trace of $T \in \mathrm{PSL}(2, \mathbb{R})$ to be $Tr(T) = |a+d|$ (why not just $a+d$?), and we classify T based on its trace.

If $Tr(T) < 2$, T fixes two complex values (which are conjugate), so it fixes exactly one value in $\mathbb{H}$. If $Tr(T) = 2$, T fixes exactly one point in $\mathbb{R}$. If $Tr(T) > 2$, T fixes two points in $\mathbb{R}$.

The last case we have to consider is if $c = 0$. Then we have $ad - bc = ad = 1$, so $d = \frac{1}{a}$. Hence, we have $z = a^2 z - ba$. Note that "∞" works as a solution to this equation. If $a = 1$, this is the only possible solution. However, if $a \neq 1$ (we can always assume $a > 0$), we have $z = \frac{ba}{1-a^2}$ is also fixed. Note this means if $Tr(T) = 2$, we fix only ∞. If $Tr(T) = |a + \frac{1}{a}| > 2$, it fixes ∞ and another point in the reals.

We therefore classify elements of PSL(2, $\mathbb{R}$) based on their trace.

An *elliptic* element has $Tr(T) < 2$ and fixes exactly one value in $\mathbb{H}$.

A *parabolic* element has $Tr(T) = 2$ and fixes exactly one point in $\mathbb{R} \cup \{\infty\}$.

A *hyperbolic* element has $Tr(T) > 2$ and fixes two points in $\mathbb{R} \cup \{\infty\}$.

Note the identity element is like a parabolic element in terms of trace, but unlike every type in its fixed points.

4 Conjugates

Conjugates are a useful tool as we study these elements. Conjugation preserves many of the properties we are interested in, besides simplifying the objects we are studying. For example, the trace is preserved under conjugation ($Tr(C^{-1}AC) = Tr(A)$); since our three types are characterized by trace, conjugation preserves the type of an element.

Not surprisingly, conjugation also preserves the algebraic structure of a group. Suppose $G \subseteq$ PSL(2, $\mathbb{R}$). Then the subgroup $C^{-1}GC$ is isomorphic to G. We directly exhibit the homomorphism $\phi : G \to C^{-1}GC$ given by $\phi : T \mapsto C^{-1}TC$, along with its inverse $\mu : V \mapsto CVC^{-1}$.

If the conjugator is chosen from PSL(2, $\mathbb{R}$), then it preserves distances in $\mathbb{H}$ and hence will also respect the topological properties of the group its applied to. In particular, the conjugate group is discrete iff the original group is discrete.

A practical application of this is if we have a subgroup of PSL(2, $\mathbb{R}$), all of whose elements fix a given set of points, we can use conjugation to take this subgroup to another one which fixes a more convenient set of points.

For example, suppose Γ is a subgroup of PSL(2, $\mathbb{R}$) made of elliptic elements that fixes some z_0. We can then choose a conjugator $C : i \mapsto z_0$ and $C \in$ PSL(2, $\mathbb{R}$). Then for any $T \in \Gamma$, $C^{-1}TC$ takes i first to z_0, then fixes z_0, then sends z_0 back to i. Therefore, all the elements of the conjugate group $C^{-1}\Gamma C$ fix i. What does this mean? $i = \frac{ai+b}{ci+d} \Leftrightarrow -c + id = ai + b \Leftrightarrow b = -c \wedge a = d$. Therefore, we know $ad - bc = a^2 + c^2 = 1$. We suggestively write $a = \cos\theta$ and $c = \sin\theta$. Then we note all the elements of the conjugate subgroups are rotation matrices.

Similarly, if we have a subgroup made of hyperbolic elements that all preserve the two boundary points r_1 and r_2, we can choose a conjugator $C : r_1 \mapsto 0$ and $C : r_2 \mapsto \infty$. Then the conjugate subgroup will fix 0 and ∞. Therefore, $c = 0$ and $b = 0$. Hence, we have a matrix where $a = \lambda$ and $d = \lambda^{-1}$. This means $z \mapsto \lambda^2 z$.

Finally, a parabolic subgroup all fixing the same point is conjugate to a group which fixes only ∞, $z \mapsto z + b$

5 Abelian Fuchsian groups

In this section, we will show a Fuchsian group is abelian if and only if it is cyclic. The implication cyclic $\Rightarrow$ abelian is trivial.

Proposition If $TS = ST$ and T and S are in PSL(2, $\mathbb{R}$), then S maps the fixed point set of T to itself injectively.
Proof If $Tp = p$, then $STp = Sp$. Since $TS = ST$, $STp = TSp = Sp$, so T also fixes Sp. The injectivity comes from the fact that S is distance-preserving. If the fixed points of T are some non-zero distance from each other, their images after S must also be a non-zero distance away from each other.

Theorem Two non-identity elements of PSL(2, $\mathbb{R}$) commute if and only if they have the same fixed-point set.
Proof ($\Rightarrow$) Suppose the two elements, T and S, commute. Then T maps the fixed point set of S to itself injectively. Similarly, S maps the fixed point set of T to itself injectively. Hence, T and S must have the same number of fixed points. If they only have one fixed point, then T sends the fixed point of S to itself. This means T also fixes S's fixed point. Similarly, S also fixes T's fixed point. Therefore, if S and T only have one fixed point, they must have the same fixed point.

The only remaining case is if S and T have two fixed points (i.e. are hyperbolic). Then we can choose a conjugator such that the conjugate of T, $C^{-1}TC$ fixes 0 and ∞. We don't know what S fixes, but we do know its conjugate has the following form.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda^{-1} \\ c\lambda & d\lambda^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda \\ c\lambda^{-1} & d\lambda^{-1} \end{pmatrix}$$

Since these two elements commute, both these lines must be equal. In particular, $b\lambda^{-1} = b\lambda$ and $c\lambda = c\lambda^{-1}$. Since T is hyperbolic, $\lambda \neq 1$, so the only way to satisfy these conditions is if $b = c = 0$. But, this means $C^{-1}SC$

fixes 0 and ∞. Since the conjugates of T and S fix the same points, T and S must fix the same points.

($\Leftarrow$) Suppose two elements have the same fixed point set. They are then of the same type. They are also mapped by the same conjugator to one of the following 3 forms:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Each of these forms commute with other forms of their type. Since the conjugates of the elements commute, the elements themselves must also commute. ■

We are now ready to deliver the theorem promised at the outset of this section.

Theorem Every abelian Fuchsian group is cyclic.
Proof Suppose Γ is an abelian Fuchsian group. Then each element from Γ must commute with every other element of Γ. However, by the previous theorem, we know two non-identity elements of PSL(2, $\mathbb{R}$) commute iff they have the same fixed points. Therefore, all the elements from Γ must have the same fixed point set, and therefore must be of the same type. Since all the non-identity elements of Γ have the same fixed point set, we can choose a conjugator to send all of the elements to a more convenient conjugate group.

Suppose all the elements are hyperbolic. Then we can choose a conjugator which will take them all to elements which fix 0 and ∞. Each of the elements will therefore have the form

$$\begin{pmatrix} \lambda_T & 0 \\ 0 & \lambda_T^{-1} \end{pmatrix},$$

where the values on the diagonal are indexed by $T \in \Gamma$. We can take this group to a discrete subgroup of $(\mathbb{R}*, \cdot)$ by mapping the matrix above to λ_T^2. (This mapping is both an isomorphism and a continuous function.) We showed at the beginning that the only discrete subgroups of $(\mathbb{R}*, \cdot)$ are cyclic. Therefore, the conjugate group must be cyclic. Since we noted earlier that conjugation preserves algebraic structure, the original group must also be cyclic.

Suppose all the elements are parabolic. Then we can use a conjugator to send them to elements fixing ∞. Then they all have the form

$$\begin{pmatrix} 1 & x_T \\ 0 & 1 \end{pmatrix}$$

We can take these directly to a discrete subgroup of $(\mathbb{R}, +)$ by taking the element above to x_T. (This mapping is both an isomorphism and a continuous mapping.) Since the only discrete subgroups of $(\mathbb{R}, +)$ are cyclic, the conjugate group, and therefore our original group must also be cyclic.

Suppose all the elements are elliptic. Then we use a conjugator to make all the elements fix i. Then all the elements will have the form

$$\begin{pmatrix} \cos\theta_T & -\sin\theta_T \\ \sin\theta_T & \cos\theta_T \end{pmatrix}$$

We map this discrete subgroup to a discrete subgroup of $(S^1, \cdot_{\mathbb{C}})$ by taking the element above to $e^{i\theta_T}$. The only discrete subgroups of $(S^1, \cdot_{\mathbb{C}})$ are cyclic, as we showed in the beginning, so our original group must also be cyclic.

Since our group must be composed of hyperbolic, parabolic, or elliptic elements, and we have shown in each case that the group must then be cyclic, the group Γ is cyclic. ■

An example application of this theorem is the fact that no Fuchsian group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, since it would then be an abelian group which is not cyclic.