LECTURE 2.

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In this lecture, we will review some of the definitions that we learned last time and we deduce simple basic algebraic properties.

Lemma 1. *Let R be a ring. Then for any $a, b \in R$ we have*

(1) $0 \cdot a = a \cdot 0 = 0$.
(2) $-(-a) = a$.
(3) $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$.
(4) $(-a) \cdot (-b) = a \cdot b$.

Proof. (1) Since $0 = 0 + 0$, for any $a \in R$, we have $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$. Thus $0 = a \cdot 0$.
(2) By the definition of additive inverse, we have $a + (-a) = 0 = -(-a) + (-a)$. Thus $a = -(-a)$.
(3) Since $b + (-b) = 0$, we have $a \cdot (b + (-b)) = a \cdot 0 = 0$. Thus $a \cdot b + a \cdot (-b) = 0$. Therefore $a \cdot (-b) = -(a \cdot b)$. The rest are similar.

□

Last time we defined a left-inverse and a right-inverse and as an exercise you are supposed to find a ring and an element which is not invertible and it has a left-inverse. The following is related

Lemma 2. *Let R be a unital ring.*

(1) *Let $a \in R$. If $b \in R$ is a left-inverse of a and $c \in R$ is a right-inverse of a, then $b = c$.*
(2) *If $a \in U(R)$, then there is a unique $b \in R$ such that $a \cdot b = b \cdot a = 1$. It is called the (multiplicative) inverse of a and it is denoted by a^{-1}.*
(3) *If $a \in U(R)$ has a unique left-inverse, then it is invertible.*

Proof. (1) $b = b \cdot 1 = b \cdot (a \cdot c) = (b \cdot a) \cdot c = 1 \cdot c = c$.
(2) If a is invertible, then by the definition it has a left-inverse b and a right-inverse c. Hence by part 1 $b = c$, which means there is $b \in R$ such that $b \cdot a = a \cdot b = 1$.
If b_1 and b_2 satisfy the above equalities, then b_1 is a left-inverse and b_2 is a right-inverse, then again by part 1 we have $b_1 = b_2$, which proves the uniqueness.
(3) Let b be the unique left-inverse of a. Then

$$(ab - 1 + b)a = (ab)a - a + ba = a(ba) - a + 1 = a \cdot 1 - a + 1 = a - a + 1 = 1,$$

which means $ab - 1 + b$ is also a left-inverse. Since we assumed b is the only left-inverse of a, we have

$$ab - 1 + b = b.$$

Hence $ab = 1$. Thus b is also a right-inverse of a, which means a is invertible.

□

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