Hyperbolic Geometry Lecture 2

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First, we begin with a couple comments from last week. There are two matrices in $SL(2,\mathbb{R})$ that correspond with the same isometry of $\mathbb{H}$, and their coordinates differ by a sign. So we will work with $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm 1_2$ as our set of isometries of $\mathbb{H}$. Also, $PSL(2,\mathbb{R})$ is a subset of a special class of functions called conformal maps. These maps satisfy the Cauchy-Riemann equations given by $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for $u + iv = T(x+iy)$. They also have the property that their derivative is never zero and they preserve angles.

1 Hyperbolic Geometry on the Unit Disk

1.1 Introducing $\mathbb{U}$

Last week, we saw how the hyperbolic geometry can be represented as a half plane, $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$ with a metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. Today, we will also consider another representation.

Definition *(The Unit Disk)* We call $\mathbb{U} = \{z \in \mathbb{C} || z| < 1\}$ the unit disk.

There is a conformal map $f : \mathbb{H} \to \mathbb{U}$ given by $f(z) = \frac{zi+1}{z+i}$. Note that this is a bijection and maps the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$ to the boundary $\{z||z| = 1\}$ of $\mathbb{U}$. We also define a distance d_u on $\mathbb{U}$ by $d_u(z_0, z_1) = d_h(f^{-1}z_0, f^{-1}z_1)$ where d_h is the distance on $\mathbb{H}$.

Theorem 1.1 *The distance d_u can be calculated with the metric $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$*

Proof We will show this by calculating the length of a curve $\gamma : [a,b] \to \mathbb{U}$. So $l_u(\gamma) = l_h(f^{-1} \circ \gamma) = \int_a^b \frac{\left|\frac{d}{dt} f^{-1} \circ \gamma(t)\right| dt}{Im(\gamma)}$. In order to do this, we must first calculate f^{-1} and we do so by solving $w = f(z) = \frac{zi+1}{z+i}$ for z. So $wz + iw = zi + 1 \Rightarrow z(w-i) = -iw + 1 \Rightarrow z = f^{-1}(w) = \frac{-iw+1}{w-i}$. Now, we calculate the denominator of the integral expression. $Im(f^{-1}\gamma) = Im\left(\frac{-i\gamma+1}{\gamma-i}\right) = Im\left(\frac{(-i\gamma+1)(\bar{\gamma}+i)}{(\gamma+i)\overline{(\gamma+i)}}\right) = \frac{Im(-i\gamma\bar{\gamma}+\gamma+\bar{\gamma}+i)}{|\gamma-i|^2} = \frac{1-|\gamma|^2}{|\gamma-i|^2}$. And the numerator becomes $\left|\frac{d}{dt} f^{-1} \circ \gamma(t)\right| dt = |(f^{-1})' \circ \gamma(t)||\gamma'(t)|dt$ and for the sake of simplicity, we will now evaluate $\frac{d}{dz} f^{-1}(z) = \frac{d}{dz}\left(\frac{-iz=1}{z-i}\right) = \frac{-i(z-i)-(-iz+1)}{(z-i)^2} = \frac{-2}{(z-i)^2}$. So the entire integral expression now becomes $\int_a^b \frac{2|\gamma'(t)|dt}{|\gamma-i|^2} \frac{|\gamma-i|^2}{1-|\gamma|^2} = \int_a^b \frac{2|dz|^2}{1-|z|^2}$ where $z = \gamma(t)$.
$\blacksquare$

For an example, let us calculate verify the following corollary.

Corollary 1.2 *The distance between a point in* $\mathbb{U}$ *with radius* r *and the origin is* $\ln\left(\frac{r+1}{r-1}\right)$.

Proof We can choose our point to be a real positive number without any loss of generality. $d_u(0,r) = \int_0^r \frac{2dx}{1-x^2} = \int_0^r \frac{1}{1-x} - \frac{1}{1+x}dx = [-\ln|1-x|]_0^r + [-\ln|1+x|]_0^r = \ln\left(\frac{1+r}{1-r}\right)$. ∎

1.2 Two related lemmas

Lemma 1.3 *For* $z, w \in \mathbb{H}$, $\cosh(d_h(z,w)) = 1 + \frac{|z-w|^2}{2Im(z)Im(w)}$.

Proof We know $d_h(ai, bi) = \ln\frac{b}{a}$ assuming without loss of generality that $a < b$. Let's verify the formula in this case. $\cosh(d_h(ai,bi)) = \frac{b/a+a/b}{2} = \frac{b^2+a^2}{2ab} = \frac{2ab}{2ab} + \frac{b^2-2ab+a^2}{2ab} = 1 + \frac{|ai-bi|^2}{2Im(ai)Im(bi)}$. Now, we know that distance is invariant under Mobius transformations, so if the right hand side is also invariant, then we can prove the equation for any z and w by selecting a Mobius transformation T mapping both of them to the imaginary axis. First let's evaluate $|Tz - Tw|^2 = \left|\frac{az+b}{cz+d} - \frac{aw+b}{cw+d}\right|^2 = \left|\frac{adz+bcw-bcz-adw}{(cz+d)(cw+d)}\right|^2 = \frac{|z-w|^2}{|cz+d|^2|cw+d|^2}$. At this point, we only have to evaluate $Im(Tz) = Im\left(\frac{az+b}{cz+d}\right) = Im\left(\frac{(az+b)(c\bar{z}+d)}{(cz+d)(c\bar{z}+d)}\right) = \frac{Im(abz\bar{z}+adz+ad\bar{z}+bd)}{|cz+d|^2} = \frac{Im(adz+bc\bar{z})}{|cz+d|^2} = \frac{(ad-bc)Im(z)}{|cz+d|^2} = \frac{Im(z)}{|cz+d|^2}$. We now see that we will get proper cancellation and that the right side of our equation is also invariant. ∎

Lemma 1.4 *In* $\mathbb{U}$, *lemma (1.3) becomes* $\cosh(d_u(z,w)) = 1 + \frac{2|z-w|^2}{(1-|z|^2)(1-|w|^2)}$.

Proof The proof of this lemma is similar to the proof of lemma (1.3). It is not difficult to show that $f^{-1}(z) = \frac{-iz+1}{z-i}$. From lemma (1.3), $\cosh(d_u(z,w)) = \cosh(d_h(f^{-1}z, f^{-1}w)) = 1 + \frac{|f^{-1}z-f^{-1}w|^2}{2Im(f^{-1}z)Im(f^{-1}w)}$. We can evaluate the numerator of this expression: $|f^{-1}z - f^{-1}w|^2 = \left|\frac{-iz+1}{z-i} - \frac{-iw+1}{w-i}\right|^2 = \left|\frac{2w-2z}{(z-i)(w-i)}\right|^2 = \frac{4|z-w|^2}{|z-i|^2|w-i|^2}$. At this point, we only have to evaluate $Im(f^{-1}z)$ to ensure that we will get proper cancellation in our fraction. Let's do that. $Im(f^{-1}z) = Im\left(\frac{-iz+1}{z-i}\right) = Im\left(\frac{(-iz+1)(\bar{z}+i)}{(z-1)(z-i)}\right) = \frac{Im(-iz\bar{z}+z+\bar{z}+i)}{|z-i|^2} = \frac{1-|z|^2}{|z-i|^2}$, which is what we want. ∎

2 Hyperbolic Trigonometry

2.1 Angle of Parallelism

Consider a triangle Δ in $\mathbb{H}$ with a vertex at ∞ and another vertex and right angle at i.

Definition We define $\Pi(a)$ to be the third angle of Δ with a side of length a opposite the vertex at ∞.

We can now explore the following relations.

Theorem 2.1 *Given Δ satisfying the conditions mentioned above, the following three relations hold.*

$$(i)\tan\Pi(a)=\frac{1}{\sinh(a)}$$

$$(ii)\sin\Pi(a)=\frac{1}{\cosh(a)}$$

$$(iii)\sec\Pi(a)=\frac{1}{\tanh(a)}$$

Proof From lemma (1.3), let the third vertex of Δ be v so we can validate (ii). $\cosh(a)=1+\frac{|i-v|^2}{2Im(i)Im(v)}=1+\frac{4\sin^2\left(\frac{\pi}{4}-\frac{\Pi(a)}{2}\right)}{2\sin\Pi(a)}=\frac{\sin\Pi(a)+\left(1-\cos\left(\frac{\pi}{2}-\Pi(a)\right)\right)}{\sin\Pi(a)}=\frac{1}{\sin\Pi(a)}$.

To validate (iii), note $\frac{1}{\sec\Pi(a)}=\cos\Pi(a)=\sqrt{1-\sin^2\Pi(a)}=\sqrt{1-\frac{1}{\cosh^2(a)}}=\frac{\sqrt{\cosh^2(a)-1}}{\cosh(a)}=\frac{\sinh(a)}{\cosh(a)}=\tanh(a)$.

Now, (i) is trivial. ∎

2.2 The Sine and Cosine Rules

In our new hyperbolic trigonometry, there are some formulae that are similar to those that we have seen in trigonometry. As usual in trigonometry, we concern ourselves with the study of triangles with sides measuring a, b, and c, and having opposite angles α, β, and γ respectively.

Theorem 2.2 (Cosine Rule I) $\cosh(c)=\cosh(a)\cosh(b)-\sinh(a)\sinh(b)\cos(\gamma)$

Proof For this proof, we will define a triangle Δ in $\mathbb{U}$ with vertices $v_c=0$, $r=Re(v_a)$ and $Im(v_a)=0$, and $z=v_b$ and opposite lengths c, a, and b respectively. Note that there is no loss of generality here. From lemma (1.4), $\cosh(c)=\frac{2|v_a-v_b|^2}{(1-|v_a|^2)(1-|v_b|^2)}+1$. We will now verify the right side matches this expression. Note using corollary (1.2), $\cosh(b)=\cosh(d_u(0,r))=\frac{\frac{1+r}{1-r}+\frac{1-r}{1+r}}{2}=\frac{2(1+r^2)}{2(1-r^2)}=\frac{1+r^2}{1-r^2}$. Now we can also find $\sinh(b)=\sqrt{\cosh^2(b)-1}=\sqrt{\left(\frac{1+r^2}{1-r^2}\right)^2-1}=\frac{\sqrt{(1+r^2)^2-(1-r^2)^2}}{1-r^2}=\frac{2r}{1-r^2}$. And from the traditional cosine rule, $|z-r|^2=|z|^2+r^2-2|z|r\cos(\gamma)$. Now, we can directly calculate the right side. $\cosh(a)\cosh(b)-\sinh(a)\sinh(b)\cos(\gamma)=\frac{1+r^2}{1-r^2}\frac{1+|z|^2}{1-|z|^2}-\frac{4r|z|}{(1-r^2)(1-|z|^2)}\frac{r^2+|z|^2-|z-r|^2}{2r|z|}$
$=\frac{(1+r^2)(1+|z|^2)-2r^2-2|z|^2+2|z-r|^2}{(1-r^2)(1-|z|^2)}=\frac{r^2|z|^2-r^2-|z|^2+1+2|z-r|^2}{(1-r^2)(1-|z|^2)}=1+\frac{2|z-r|^2}{(1-r^2)(1-|z|^2)}$, which is what we wanted. ∎

Theorem 2.3 (Sine Rule) $\frac{\sinh(a)}{\sin(\alpha)}=\frac{\sinh(b)}{\sin(\beta)}$

Proof $\left(\frac{\sinh(c)}{\sin(\gamma)}\right)^2=\frac{\sinh^2(c)}{1-\cos^2(\gamma)}=\frac{\sinh^2(c)}{1-\left(\frac{\cosh(a)\cosh(b)-\cosh(c)}{\sinh(a)\sinh(b)}\right)^2}$ from theorem (2.2).

We will now show the right side is symmetric in a, b, and c. It can be rewritten as $\frac{\sinh^2(a)\sinh^2(b)\sinh^2(c)}{\sinh^2(a)\sinh^2(b)-(\cosh(a)\cosh(b)-\cosh(c))^2}$, which is symmetric provided the denominator is symmetric. Using $\cosh^2(x)=1+\sinh^2(x)$ and expanding, we

see the denominator is equivalent to
$\sinh^2(a)\sinh^2(b) - \cosh^2(a)\cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c) =$
$\sinh^2(a)\sinh^2(b) - (1+\sinh^2(a))\cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c) =$
$\sinh^2(a)(\sinh^2(b) - \cosh^2(b)) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c)$.
Since $\sinh^2(b) - \cosh^2(b) = -1$, we are done. ∎

Theorem 2.4 (Cosine Rule II) $\cosh(c) = \frac{\cos(\alpha)\cos(\beta)+\cos(\gamma)}{\sin(\alpha)\sin(\beta)}$

Proof This proof is rather straightforward and tedious. It is therefore left as an exercise. ∎

Note that this third theorem has no analogue in trigonometry on the Euclidean plane. Can you see why? It determines the length of a side from the three angles, whereas in Euclidean trigonometry, there are many different triangles with congruent angles.

3 Hyperbolic Area

Since we have explored the notion of length, and we are working in a two dimensional space, it seems only natural to explore the concept of area next.

Definition *(Area)* For a region $A \subset \mathbb{H}$, define the area to be $\mu(A) = \int_A \frac{dxdy}{y^2}$ if the integral exists.

This is a natural definition of area as we will see in the following theorem.

Theorem 3.1 *The area $\mu(A)$ is invariant under transformations in $PSL(2,\mathbb{R})$. I.e. for $T \in PSL(2,\mathbb{R})$, $\mu(T(A)) = \mu(A)$.*

Proof First define $T(z) = \frac{az+b}{cz+d}$ where $ad-bc = 1$. Let $w = u+iv = T(z)$. Then $dudv = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) dxdy = \left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial v}{\partial x}\right|^2\right) dxdy = \left|\frac{\partial}{\partial x}(u+iv)\right|^2 dxdy =$ $\left|\frac{\partial}{\partial x}Tz(x)\right|^2 dxdy = \left|\frac{dT}{dz}\right|^2 dxdy = |cz+d|^{-4}dxdy$. The second equality holds by the Cauchy-Reimann equations. Then, recalling our proof of lemma (1.3) that $v = Im(T(z)) = \frac{Im(z)}{|cz+d|^2}$, we calculate
$\mu(T(A)) = \int_{T(A)} \frac{dudv}{v^2} = \int_A \frac{|cz+d|^{-4}dxdy}{Im(z)^2|cz+d|^{-4}} = \int_A \frac{dxdy}{y^2} = \mu(A)$ ∎

4 Gauss-Bonnet Theorem

Theorem 4.1 (Gauss-Bonnet) *Let Δ be a triangle with angles α, β, and γ. Then $\mu(\Delta) = \pi - \alpha - \beta - \gamma$.*

Proof First, we will observe the case where a vertex of the triangle lies on the boundary $\partial\mathbb{H}$ and from theorem (3.1) use a mobius transformation to map this vertex to ∞. This yields a region bounded on the left and right by two vertical lines and below by a subset of a circle. Without changing angles and area, we may apply transformations of the form $T(z) = z + b$ or $T(z) = az$ so that this circle is the unit circle. (Let the left side be adjacent to angle α and have horizontal coordinate a, while the right side is adjacent to angle β and has horizontal coordinate b.) Finally, we have a region over which it is not difficult

to integrate.
$\mu(A) = \int_A \frac{dxdy}{y^2} = \int_a^b dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} = \int_a^b \frac{dx}{\sqrt{1-x^2}}$ With a change of coordinates ($x = \cos(\theta)$), the last integral becomes $\int_{\pi-\alpha}^{\beta} \frac{-\sin(\theta)}{\sin(\theta)} = \pi - \alpha - \beta$.
Note that in this case, $\gamma = 0$. We are now left with the case where no vertex lies on $\partial\mathbb{H}$. To handle this, we simply reduce this to two instances of our previous case by extending one of the line segments to a ray. We form two new triangles and take their difference to be the area of the original triangle. $\blacksquare$

5 Various Trigonometries

We will end this lecture with a few comments on trigonometries in various different geometries. Those are the spherical geometry, the hyperbolic geometry and the Euclidean geometry. In all three cases, we deal with triangles with sides measuring a, b, and c, and having opposite angles α, β, and γ respectively.

5.1 Triangles on the Sphere

For a triangle on a sphere with radius r, the three trigonometric identities take the following form.

$$\text{Sine Rule: } \frac{\sin(a/r)}{\sin(\alpha)} = \frac{\sin(b/r)}{\sin(\beta)} \tag{1}$$

$$\text{Cosine Rule I: } \cos(c/r) = \cos(a/r)\cos(b/r) + \sin(a/r)\sin(b/r)\cos(\gamma) \tag{2}$$

$$\text{Cosine Rule II: } \cos(c/r) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)} \tag{3}$$

5.2 Triangles on the Hyperbolic Plane

To convert the previous equations into Hyperbolic form, we use the following definition. Informally, it may be thought of as the "sphere with imaginary radius."

Definition Let $\mathbb{H}_r$ be the upper half plane with metric $ds^2 = \frac{r^2(dx^2+dy^2)}{y^2}$.

Note that letting $r = 1$, we get the usual hyperbolic plane. We now take the three previous equations and replace r with ir. Using the identities $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$, we obtain the following hyperbolic trigonometric identities.

$$\text{Sine Rule: } \frac{\sin(a/r)}{\sin(\alpha)} = \frac{\sin(b/r)}{\sin(\beta)} \tag{4}$$

$$\text{Cosine Rule I: } \cosh(c/r) = \cosh(a/r)\cosh(b/r) + \sinh(a/r)\sinh(b/r)\cos(\gamma) \tag{5}$$

$$\text{Cosine Rule II: } \cosh(c/r) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)} \tag{6}$$

5.3 Trigonometry on the Euclidean Plane

We can convert the above equations to Euclidean trigonometric equations by taking the second order Taylor expansion of sin, cos, sinh, and cosh and letting $a, b, c, r \to \infty$. Equation (4) becomes

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} \tag{7}$$

which is the standard Sine Rule. Likewise, equation (5) becomes

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \tag{8}$$

which is the standard Cosine Rule. As mentioned earlier, there is no analogue to the Cosine Rule II.