$X = \mathbb{R}^2$, $\quad pRq \iff p-q \in \mathbb{Z}^2$.

$Y = [0,1]\times[0,1]$, The relation S is defined as follows

I. $\forall y \in Y$, ySy

II. $\forall x \in [0,1]$, $\quad (0,x)\, S\, (1,x) \wedge (1,x)\, S\, (0,x)$

III. $\forall x \in [0,1]$, $\quad (x,0)\, S\, (x,1) \wedge (x,1)\, S\, (x,0)$

IV. $(0,0)\, S\, (1,1) \wedge (1,1)\, S\, (0,0) \wedge (1,0)\, S\, (0,1) \wedge (0,1)\, S\, (1,0)$.

R is an equivalence relation.

reflexive. $\quad \forall p \in X$, $\quad p-p = (0,0) \in \mathbb{Z}^2 \Rightarrow pRp$.

symmetric. $\quad pRq \Rightarrow p-q \in \mathbb{Z}^2 \Rightarrow q-p \in \mathbb{Z}^2 \Rightarrow qRp$.

transitive. $\left.\begin{array}{l} pRq \Rightarrow p-q \in \mathbb{Z}^2 \\ qRs \Rightarrow q-s \in \mathbb{Z}^2 \end{array}\right\} \Rightarrow p-s = (p-q)+(q-s) \in \mathbb{Z}^2$.

$\Rightarrow pRs$ ■

S is an equivalence relation.

reflexive. By the condition I, we have $\forall y \in Y$, ySy.

symmetric & transitive In order to address these properties, we reformulate S:

Claim. $\forall y_1, y_2 \in Y$, $y_1 S y_2 \iff y_1 - y_2 \in \mathbb{Z}^2$.

Pf of claim. $(\Rightarrow)$

If y_1, y_2 satisfy Ⓘ, $y_1 - y_2 = (0,0)$.

" " " " Ⓘⓘ, $y_1 - y_2 = (\pm 1, 0)$.

" " " " Ⓘⓘⓘ, $y_1 - y_2 = (0, \pm 1)$.

" " " " Ⓘⓥ, $y_1 - y_2 \in \{\pm(1,1), \pm(1,-1)\}$.

Hence $y_1 S y_2 \Rightarrow y_1 - y_2 \in \mathbb{Z}^2$.

$(\Leftarrow)$ $y_1 = (a_1, b_1), y_2 = (a_2, b_2) \in Y$ and $y_1 - y_2 \in \mathbb{Z}^2$

On the other hand, if $0 \le x_1, x_2 \le 1$ and $x_1 - x_2 \in \mathbb{Z}$,

then $x_1 - x_2 = 0 \vee (x_1 = 0 \wedge x_2 = 1) \vee (x_1 = 1 \wedge x_2 = 0)$

Thus $[a_1 = a_2 \vee (a_1 = 0 \wedge a_2 = 1) \vee (a_1 = 1 \wedge a_2 = 0)]$

$\wedge [b_1 = b_2 \vee (b_1 = 0 \wedge b_2 = 1) \vee (b_1 = 1 \wedge b_2 = 0)]$

Therefore $(y_1 = y_2) \vee$

$(y_1 = (a_1, 0) \wedge y_2 = (a_1, 1)) \vee (y_1 = (a_1, 1) \wedge y_2 = (a_1, 0)) \vee$

$(y_1 = (0, b_1) \wedge y_2 = (1, b_1)) \vee (y_1 = (1, b_1) \wedge y_2 = (0, b_1)) \vee$

$(y_1 = (0,0) \wedge y_2 = (1,1)) \vee (y_1 = (0,1) \wedge y_2 = (1,0)) \vee$

$(y_1 = (1,0) \wedge y_2 = (0,1)) \vee (y_1 = (1,1) \wedge y_2 = (0,0))$;

which is equivalent to say $y_1 S y_2$. ■

Now similar to the relation R, one can show S is symmetric and transitive.

There is a bijection between $Y/_S$ and $X/_R$.

Pf.: Let $f: Y/_S \longrightarrow X/_R$.

$$f([y]_S) = [y]_R.$$

Well-defined. $y_1, y_2 \in Y$ and $y_1 S y_2 \Rightarrow$ by the above claim, $y_1 - y_2 \in \mathbb{Z}^2 \Rightarrow y_1 R y_2 \Rightarrow [y_1]_R = [y_2]_R$

1-1. $f([y_1]_S) = f([y_2]_S) \Rightarrow [y_1]_R = [y_2]_R$

$$\Rightarrow y_1 R y_2 \Rightarrow y_1 - y_2 \in \mathbb{Z}^2$$

$$\overset{\text{by the above claim}}{\Longrightarrow} y_1 S y_2$$

$$\Rightarrow [y_1]_S = [y_2]_S.$$

onto. $\forall\, x = (a,b) \in X$ there integer numbers n and m such that $y := x - (n,m) = (a-n, b-m) \in Y$.
Hence $y R x$. Thus

$$f([y]_S) = [y]_R = [x]_R. \quad ■$$