1. Let p be an odd prime, and let G be a group of order $p(p+1)$.

ⓐ Prove that G is not simple.

ⓑ Prove that, if G has more than one Sylow p-subgroup, then $p = 2^k - 1$ for some $k \in \mathbb{Z}^+$.

[Hint: Step 1. $n_p = 1$ or $n_p = p+1$.

From this point on we will assume $n_p = p+1$ and $\mathrm{Syl}_p(G) = \{P_1, \dots, P_{p+1}\}$.

Let $H := (G \setminus (P_1 \cup \cdots \cup P_{p+1})) \cup \{1\}$.

Step 2. Show H is invariant under conjugacy. and $|H| = p+1$.

Step 3. Prove that $\forall i$, $N(P_i) = P_i = C(P_i)$ and conclude that

$$\forall h \in H \setminus \{1\},\ C_G(h) \subseteq H.$$

Step 4. Using Step 3 and the fact that

$\forall h \in H \setminus \{1\}$, $Cl(h)$ (conjugacy class) $\subseteq H \setminus \{1\}$, prove that ⓐ $C_G(h) = H$.

ⓑ $Cl(h) = H \setminus \{1\}$.

Step 5. $\boxed{4a}$ implies $H \lhd G$, and $\boxed{4b}$ implies all the non-identity elements of H have the same order.

Step 6 Since $2 \mid p+1 = |H|$, conclude that H is a 2-gp, and so $2^k = p+1$.]

Remark. In fact the above argument proves that, if $n_p \neq 1$, then $G \simeq \mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})^k$ where $p = 2^k - 1$ and $\mathbb{Z}/p\mathbb{Z} \hookrightarrow GL_k(\mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^k)$

2. ⓐ Prove that S_6 has a subgroup isomorphic to S_5 which acts transitively on $\{1,2,3,4,5,6\}$ via the natural action of S_6.

[Hint. Step 1. Compute $|Syl_5(S_5)|$.

Step 2. Show that the action of S_5 on $Syl_5(S_5)$ gives us an embedding of S_5 into $S_{Syl_5(S_5)}$.

<u>Step 3</u>. Show why <u>step 2</u> is enough to get the desired result.]

(b) Show that $Aut(S_6) \neq Inn(S_6)$, i.e. there is an automorphism of S_6 which is not an inner automorphism.

[<u>Hint</u>. Let $H \leq S_6$ be the subgroup that part (a) gives us, i.e.
- $H \simeq S_5$
- $H \curvearrowright \{1,2,3,4,5,6\}$ transitively.

<u>Step 1</u> Consider $S_6 \curvearrowright S_6/H$. Show that this action induces an automorphism $\theta : S_6 \to S_6$.

<u>Step 2</u> Show that $\theta(H) \curvearrowright \{1,\dots,6\}$ has a fixed point.

<u>Step 3</u> Show that, if Ψ is an inner automorphism, then $\Psi(H) \curvearrowright \{1,\dots,6\}$ transitively.]

<u>Remark</u>. • The group of outer automorphism $Out(G)$

is $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. A group is called complete if ① $\mathrm{Out}(G)=\{1\}$ ② $Z(G)=\{1\}$.

- Here you prove that $\mathrm{Out}(S_6)\neq\{1\}$.
- It is proved that S_n is a complete group if and only if $n\neq 2$ or 6.

3. ⓐ Suppose $n\in\mathbb{Z}^{\geq 2}$ and $\gcd(\phi(n),n)=1$.

Prove that any group of order n is cyclic.

[Recall: $\phi(\prod p_i^{m_i})=\prod p_i^{m_i-1}(p_i-1)$ where p_i's are distinct primes.]

[Hint: Proceed by induction on n.

Step 1. n is square-free; $m\mid n \Rightarrow \gcd(m,\phi(m))=1$; Conclude that by the induction hypothesis all the proper subgroups are cyclic.

Step 2. It is enough to show $G=Z(G)$.

Step 3. Consider $G/Z(G)$ to show that it is enough to show $1\neq Z(G)$.

Step 4. Prove that if $1 \neq N \trianglelefteq G$, then $N \subseteq Z(G)$.

(Use: • N: abelian $\Rightarrow$ $G/N \curvearrowright N$.

• $\gcd(|G/N|, |Aut(N)|) = 1 \Rightarrow Hom(G/N, Aut(N)) = 1$)

Step 5. Prove that if $M_1 \neq M_2$ are maximal subgroups of G, then $M_1 \cap M_2 = \{1\}$.

(Use: $x \in M_1 \cap M_2 \Rightarrow C_G(x) \supseteq M_1 \cup M_2$.)

Step 6. $\exists$ two maximal subgroups that are not conjugate of each other. And show that

$$\left| \bigcup_{g \in G} g M_1 g^{-1} \cup \bigcup_{g \in G} g M_2 g^{-1} \right| > |G|$$

to get a contradiction.]

(b) Suppose $n \in \mathbb{Z}^{\geq 2}$ and $\gcd(\phi(n), n) \neq 1$. Prove that there is a non-cyclic group of order n.

4. Let G be a group of order 144. Prove that G is not simple.

5. Show that the only simple group of order 60 is A_5.

[Hint: Step 1. Suppose G is a non-abelian simple gp and $H \leq G$. Then $|G| \mid |H|!/2$.

In particular, if $|G| = 60$ and G is simple, then G has no subgroup of index ≤ 4.

Step 2. Show that $n_5 = 6$, $n_3 = 10$, and $n_2 = 5$ or 15.

Step 3. Show that, if $n_2 = 5$, we are done.

Step 4. If $n_2 = 15$, by a counting argument show that there are two Sylow 2-Subgroups P and Q s.t.

$$|P \cap Q| = 2.$$

So $\langle P, Q \rangle \subseteq N_G(P \cap Q) \subsetneqq G$.

Step 5. Conclude that $[G : N_G(P \cap Q)] = 5$ and, as in step 3, show $G \simeq A_5$.]

Remark. In fact, step 5 gives us a contradiction as A_5 has only 5 Sylow 2-subgroups.

Disclaimer: There might be easier and/or nicer solutio
that are not based on the given hints!