1. Let G be a finite group and X be a finite set. Assume $G \curvearrowright X$. For any $g \in G$, let X^g be the set of fixed points of g, i.e.

$$X^g := \{x \in X \mid g \cdot x = x\}.$$

Prove that $\left| {}_{G}\backslash^{X} \right| = \mathbb{E}(|X^g|)$.

($\mathbb{E}(|X^g|)$ is the average of $|X^g|$, i.e. $\frac{1}{|G|} \sum_{g \in G} |X^g|$.)

Hint. • Consider $S = \{(g, x) \in G \times X \mid g \cdot x = x\}$.

$$\Rightarrow |S| = \sum_{g \in G} |X^g| = \sum_{x \in X} |G_x|.$$

• $G_{g \cdot x} = g G_x g^{-1}$.

2. Let G be a finite group and X be a finite set. Assume $G \curvearrowright X$ transitively and $|X| \neq 1$. Prove that there is $g \in G$ with no fixed points.

Hint. • Use 1 and notice that $X^{id.} = X$.

3(a). Let G be a finite group and $H \lneq G$. Prove that

$$\bigcup_{g\in G} gHg^{-1} \neq G.$$

(b) Does part (a) hold for an infinite group?

Hint. (a) There are various ways to solve this problem. Here is one method: assume the contrary and get a contradiction using Problem 2.

(b) Think about $G = GL_2(\mathbb{C})$ and $H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \, ac \neq 0 \right\}$.

4. For any permutation $\sigma \in S_n$, let m_σ be the number of fixed points of σ, i.e.

$$m_\sigma := \left| \{ 1 \leq i \leq n \mid \sigma(i) = i \} \right|.$$

Prove that $\displaystyle\sum_{\sigma \in S_n} m_\sigma = n!$

5. $M, N \trianglelefteq G$ and $M \cap N = \{1_G\} \Rightarrow \forall m \in M, n \in N, mn = nm.$

6. Let G be a finite group. Let p be the smallest prime factor of G. Assume $H \leq G$ and $[G:H] = p$.

Prove that $H \triangleleft G$.

__Hint.__ Let N be the core of H. Then as we showed in class we have $G/N \hookrightarrow S_p$. Now use the assumption to deduce $|G/N| = p$.

7. Let G be a group, $H \leq G$, $N \trianglelefteq G$.

   (a) Assume $|H| < \infty$, $[G:N] < \infty$ and $(|H|, [G:N]) = 1$.

   Prove that $H \leq N$.

   (b) Assume $|N| < \infty$, $[G:H] < \infty$ and $(|N|, [G:H]) = 1$.

   Prove that $N \leq H$.

8. Let G be a finite non-abelian simple group.

   Assume $\exists\ H \lneq G$, $[G:H] = n$. Then

$$G \hookrightarrow A_n.$$