**FINAL: MATH 109, FALL 2011.**

No books, notes, calculators, cell phones, laptops, tablets, ... are permitted. Good luck !

(1) (10 points) Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $a|bc$, then $a|c$.

(2) (a) (3 points) Let $k, l \in \mathbb{N}$. Prove that $2^k - 1 | 2^l - 1$ if $k|l$.

*Date*: 12/5/2011.

(b) (12 points) Prove that $\gcd(2^n - 1, 2^m - 1) = 2^{\gcd(n,m)} - 1$ for any $m, n \in \mathbb{N}$.

(3) (15 points) For any set $X$, prove that $|X| < |P(X)|$ where $P(X)$ is the power set of $X$.

(4) (10 points) Find all the solutions of 106x=6 in $\mathbb{Z}/58\mathbb{Z}$.

(5) (10 points) Let $X = \{1, 2, 3, 4\}$. How many relations $R$ are there on $X$ which satisfy all these conditions: $R$ is reflexive. $R$ is symmetric, and $1 R\, 3$? Explain your solution.

(6) (10 points) Let $X = \{1, 2, \ldots, 2011\}$. Compute $|\{A \subseteq X \mid |A| \text{ is even}\}|$. Explain your solution.

(7) (15 points) Let $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $f([x]_m) = [12x]_n$. Prove that $f$ is well-defined if and only if $n/\gcd(12, n)$ divides $m$. What is the necessary and sufficient condition under which $f$ is a bijection?

(8) (10 points) Find the remainder of the division of $3^{2011}$ by 2012.