Math 109 Midterm 2 Review

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1 Strong induction.

1. Let $a_0 = 2$, $a_1 = 4$ and $a_{n+1} := 4a_n - a_{n-1}$ for any integer $n \in \mathbb{Z}^+$.
   (a) By strong induction on n prove that $a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$.
   (b) Prove that $a_n = \lfloor (2+\sqrt{3})^n \rfloor$ for any positive integer n, where $\lfloor x \rfloor$ is the integer part of x. (Hint: use the first part and the fact that $0 < 2-\sqrt{3} < 1$.)

2. Prove that every amount of postage of 12 cents or more can be achieved using only 4-cent and 5-cent stamps.

3. Consider a chess board of size $2^n \times 2^n$ for some positive integer n, with a single corner cell removed. Suppose you have only L-shaped playing pieces which cover three cells of the board:  . You are free to orient these pieces in any way. Use induction to show that for any positive integer n you can cover the board with these L-shaped pieces.

4. Do the bonus problem, number 2 from Homework 6. That is, show that
   (a) if G_1 is P and G_2 is N, then $G_1 \oplus G_2$ is P, and
   (b) if G_1 and G_2 are N, then $G_1 \oplus G_2$ is N.

   Hint: Prove both parts at the same time, by strong induction on $l(G_1) + l(G_2)$.

5. There are two boxes. Initially, one box contains m chips and the other contains n chips. Such a position is denoted by (m, n), where $m > 0$ and $n > 0$. The two players alternate moving. A move consists of emptying one of the boxes, and dividing the contents of the other between the two boxes with at least one chip in each box. There is a unique terminal position, namely $(1, 1)$. Last player to move wins. Assume that both players do the best possible move. Prove that the second player can win if and only if both m and n are odd. (Hint: use strong induction on $m + n$.)

2 Language of set theory.

1. List all subsets of $\{1, 2, \{1\}, \{1, 2\}\}$.

2. Show that if $(A \cup C) \subseteq (A \cup B)$ and $(A \cap C) \subseteq (A \cap B)$, then $C \subseteq B$.

3. Define the complement of a subset A of X as $A^C = \{a \in X |\ a \notin A\}$. Let A, B be two subsets of X. Prove the following De Morgan's Laws for sets.
   (a) $(A \cap B)^C = A^C \cup B^C$
   (b) $(A \cup B)^C = A^C \cap B^C$

4. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

5. Let A and B be non-empty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.

6. Let $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

3 Quantifiers.

1. Prove or disprove: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $\forall z \in \mathbb{R}, x + y = z$.
2. Show that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ with $n \geq N$,
$$\frac{n}{n^2 - 1} < \epsilon.$$
3. Prove that
$$\forall \varepsilon > 0, \exists\, \delta > 0,\ 0 < |x - 1| < \delta \Rightarrow |x^2 - 1| < \varepsilon.$$
4. For each part, prove or disprove the statement.
   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, -x^4 < y$.
   (b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, -x^4 < y$.
   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, -x^3 < y$.
   (d) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, -x^3 < y$.
5. An integer $n > 1$ is called a 2-almost prime if it is product of at most two prime factors. For instance $2, 3, 5, 6, 7, 9, 10$ are 2-almost primes, but $8, 12, 16, 18$ are NOT 2-almost primes. Let $\mathcal{P}_2$ be the set of all the 2-almost primes. Prove that
$$\forall n \in \mathbb{Z}^{>1}, n \notin \mathcal{P}_2 \Rightarrow \exists\, m_1, m_2, m_3 \in \mathbb{Z}^{>1}, n = m_1 \cdot m_2 \cdot m_3.$$
6. Suppose that $A \subseteq \mathbb{Z}$. Write the following statement entirely in symbols using quantifiers. Then write the negative of the statement using symbols. Finally, give an example of a set A for which the statement is true, and an example of a set A for which the statement is false. The statement is:

   "There is a greatest number in the set A."

4 Functions.

1. Define functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and $g(x) = x^2 - 1$.
   (a) Find the functions $f \circ f, f \circ g, g \circ f$, and $g \circ g$.
   (b) Suppose we changed the codomain of f to be $\mathbb{R}^{\geq 0}$. Which of the functions in part (a) would still be defined?
   (c) List all elements of the set $\{x \in \mathbb{R} |\ f(g(x)) = g(f(x))\}$.
2. Given $Y \subseteq Z$, define the characteristic function of Y as the function $\chi_Y : Z \to \{0, 1\}$ where
$$\chi(z) = \begin{cases} 1 & \text{if } z \in Y \\ 0 & \text{if } z \notin Y \end{cases}$$

   Suppose that A and B are subsets of Z.

   (a) Prove that $\chi_{A \cap B} = \chi_A(z)\chi_B(z)$. That is, prove that the characteristic function of the intersection $A \cap B$ is the product of the characteristic functions of A and B.
   (b) Find a subset $C \subset Z$ whose characteristic function is given by $\chi_C = \chi_A(z) + \chi_B(z) - \chi_A(z)\chi_B(z)$.