Mathematics 103B Practice problems
Exam 2

1. Let \( f(x) = x^4 + 4x + 5 \) and \( g(x) = x^2 + 2x + 2 \) be polynomials in \( \mathbb{Z}/7\mathbb{Z}[x] \). Find the quotient and remainder for dividing \( f(x) \) by \( g(x) \).

2. Give an explicit example of an infinite field of characteristic 7. Is there an example of an infinite field of characteristic 6?

3. Let \( \mathbb{Z}[x, y] = \mathbb{Z}[x][y] \) be the polynomials in the two variables \( x, y \) with integer coefficients. Is there a field that contains \( \mathbb{Z}[x, y] \)? How about \( \mathbb{Z}/6\mathbb{Z}[x, y] \); is there a field that contains this ring?

4. Let \( F \) be a field of characteristic \( p \). We proved in class that \( F \) contains the finite field \( \mathbb{Z}/p\mathbb{Z} \) as a subring. Suppose \( a \in F \) satisfies \( a^p = a \). Prove that \( a \) is in the subfield \( \mathbb{Z}/p\mathbb{Z} \).

5. Set \( R = \mathbb{Z}/10\mathbb{Z} \). Give an example of an ideal of \( R \) that is prime. Be sure to prove that your ideal is prime. Is your ideal also maximal?

6. Are there any examples of ideals of the ring \( R = \mathbb{Z}[i] \) that are prime but maximal?

7. Suppose \( R \) is a commutative ring with unity, and \( I \subseteq R \) is a prime ideal so that \( R/I \) has finite size. Prove that \( I \) is maximal. Give an example of a commutative ring \( R \) with unity and an ideal \( I \) so that
   - \( R \) has infinite size and
   - \( R/I \) has finite size but \( I \) is not maximal.

8. Let \( R = \mathbb{Z}[i] \). Find a prime number \( p \) so that \( R/\langle p \rangle \) is not an integral domain. Be sure to prove your answer.

9. For each of the following polynomials, factor it into a product of irreducibles. If the polynomial itself is irreducible, then say so. (Be sure to prove your answers.)
   (a) \( x^2 + 5x + 3 \in \mathbb{Z}/2\mathbb{Z}[x] \).
   (b) \( x^3 + x^2 + x + 1 \in \mathbb{Q}[x] \). **Hint**: Observe that \(-1\) is a root of this polynomial.
   (c) \( x^4 + 27x^2 + 6 \in \mathbb{Q}[x] \).
   (d) \( x^3 - x + 1 \in \mathbb{Z}/7\mathbb{Z}[x] \).
   (e) \( x^3 + 2x + 1 \in \mathbb{Q}[x] \). **Hint**: Consider the same polynomial in \( \mathbb{Z}/3\mathbb{Z}[x] \).

10. Construct explicitly a field of size 49.

11. Is \( \mathbb{Z}[^5]/(1 + \sqrt{5}) \) a field? How many elements are in this quotient?

12. Let \( R \) be the ring \( R = \mathbb{Z}[x]/(x^3 + 4x + 1) \) and let \( \alpha \) denote the image of \( x \) in \( R \). What is the size of the ring \( R/(\alpha - 1) \)?

13. Give an example of a ring \( R \) that
• Contains the complex numbers $\mathbb{C}$ as a subring;
• As a $\mathbb{C}$ vector space, is of dimension four.

14. Prove that the rings $\mathbb{Z}[x]/\langle x^2 \rangle$ and $\mathbb{Z} \times \mathbb{Z}$ are not isomorphic.

15. Consider the ring homomorphism $\phi : \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ defined by $\phi(x) = 1 + \sqrt{2}$. Find a polynomial $p(x)$ so that $\ker(\phi) = \langle p(x) \rangle$. 