

Mathematics 204A Homework 8

Due: Friday 8 December 2023

1. (Murty and Esmonde, exercise 5.6.14) Let $K = \mathbf{Q}(\theta)$ be a degree n extension of $\mathbf{Q}$, and assume $\theta \in \mathcal{O}_K$. Suppose p divides $\text{disc}(1, \theta, \dots, \theta^{n-1})$ but p^2 does not divide this quantity. Prove that $p|d_K$ and thus that p ramifies in K .
2. (Murty and Esmonde, exercise 5.6.15) Let K be an algebraic number field of discriminant d_K . Show that the normal closure of K contains $\mathbf{Q}(\sqrt{d_K})$.
3. Suppose N over K is a Galois extension of number fields with Galois group G , and $L = N^H$ is the fixed field of a subgroup H of G . Let $\mathfrak{p}$ be a prime of K , $\mathfrak{P}$ a prime of N over $\mathfrak{p}$, and $I_{\mathfrak{P}}$ the inertia group of $\mathfrak{P}$.
 - (a) Check that the inertia group of $\mathfrak{P}$ over $\mathfrak{P}_L := \mathfrak{P} \cap L$ is $I_{\mathfrak{P}} \cap H$.
 - (b) Let e_1 be the ramification index of $\mathfrak{P}$ over $\mathfrak{P}_L$ and e_2 the ramification index of $\mathfrak{P}_L$ over $\mathfrak{p}$. Deduce that $e_1 = |I_{\mathfrak{P}} \cap H|$ and $e_2 = [I_{\mathfrak{P}} : I_{\mathfrak{P}} \cap H]$.
 - (c) Conclude that if $\mathfrak{P}_L$ is unramified over $\mathfrak{p}$, then $I_{\mathfrak{P}} \subseteq H$.
4. (Murty and Esmonde, exercise 5.6.28) Let L/K be a finite extension of algebraic number fields. If $\tilde{L}$ denotes the normal closure, show that a prime $\mathfrak{p}$ of $\mathcal{O}_K$ is unramified in L if and only if it is unramified in $\tilde{L}$. (Hint: Consider inertia groups and use the previous exercise.)