

Mathematics 204A Homework 7
Due: Monday 27 November 2023

1. Suppose $-d$ is a negative integer, squarefree, congruent to 2 or 3 modulo 4. Consider $K = \mathbf{Q}[\sqrt{-d}]$ and its ring of integers $\mathbf{Z}[\sqrt{-d}]$. Suppose $d = p_1 \cdots p_s$ is a product of s distinct primes.
 - (a) Let P_i be the prime ideal of $\mathbf{Z}[\sqrt{-d}]$ satisfying $P_i^2 = (p_i)$. Prove that P_i is not principal. Deduce that $[P_i]$ defines a nontrivial element in the two-torsion subgroup $Cl_K[2]$ of the class group. Hint: Consider norms.
 - (b) Prove that the class $[P_1] \cdots [P_s]$ is trivial in Cl_K .
 - (c) Suppose $\mathfrak{a} = P_{i_1} \cdots P_{i_r}$ is a product of $r < s$ distinct elements of the set $\{P_1, \dots, P_s\}$. Prove that $\mathfrak{a}$ is not principal. Hint: Again consider norms.
 - (d) Deduce that the size of the two-torsion subgroup $Cl_K[2]$ is at least 2^{s-1} .
2. (Neukirch, section 9, exercise 1) Prove that if $L|K$ is a Galois extension of algebraic number fields with noncyclic Galois group, then there are at most finitely many nonsplit prime ideals of K .
3. (Neukirch, section 9, exercise 4) Let $L|K$ be a finite (not necessarily Galois) extension of algebraic number fields, and $N|K$ the normal closure of $L|K$. Show that a prime ideal $\mathfrak{p}$ of K is totally split in L if and only if it is totally split in N . (See the hint in the text.)