

Mathematics 204A Homework 6

Due: Friday 17 November 2023

1. (Murty and Esmonde, exercise 4.5.20) Suppose K is a number field with r_1 real embeddings and $2r_2$ complex embeddings so that $r_1 + 2r_2 = [K : \mathbf{Q}] = n$. Show that the discriminant d_K has sign $(-1)^{r_2}$.
2. Show that $\mathcal{O}' = \mathbf{Z}[2^{1/3}]$ is the ring of integers in $K = \mathbf{Q}(2^{1/3})$. Hint: Compute the discriminant $\text{disc}(1, 2^{1/3}, 2^{2/3})$ using a previous homework question, and see that it is only divisible by 2 and 3. Note also by a previous homework question, 2 does not divide $[\mathcal{O}_K : \mathcal{O}']$, because $x^3 - 2$ is Eisenstein at two. By considering $(x - 1)^3 - 2$, observe that 3 also does not divide this index.
3. Find the factorizations of the ideals (2) , (3) and (5) in $\mathcal{O}_K$, where $K = \mathbf{Q}(2^{1/3})$.
4. Suppose K is a degree three extension of $\mathbf{Q}$, such that (2) splits completely. Prove that $\mathcal{O}_K$ is not of the form $\mathbf{Z}[\alpha]$. Hint: if $\mathcal{O}_K = \mathbf{Z}[x]/(x^3 + ax^2 + bx + c)$, what are the ways that 2 can decompose?