

Mathematics 204A Homework 5

Due: Friday 10 November 2023

Instructions: Please turn in solutions to 4 of the problems below.

1. (Neukirch, section 6, exercise 2 with typo fixed.) We proved in class that in every ideal class of the ring of integers $\mathcal{O}_K$ of a number field K there is a representative $\mathfrak{a}$ which is an integral ideal with norm $N(\mathfrak{a}) \leq (2/\pi)^s \sqrt{|d_K|}$, where $2s$ is the number of embeddings $\tau : K \rightarrow \mathbf{C}$ that do not land in $\mathbf{R}$. (See the proof of Theorem 6.3 of Neukirch.) Using this fact, prove that the quadratic fields with discriminants 5, 8, 12, -3, -4, -7, -8, -11 have class number 1.
2. (Neukirch, section 6, exercise 6). Let $\mathfrak{a}$ be an integral ideal of K with $\mathfrak{a}^m = (a)$. Show that $\mathfrak{a}$ becomes a principal ideal in the field $L = K(a^{1/m})$, in the sense that $\mathfrak{a}\mathcal{O}_L = (\alpha)$.
3. (Neukirch, section 6, exercise 7) Show that, for every number field K , there exists a finite extension L such that every ideal of K becomes a principal ideal.
4. (Neukirch, section 5, exercise 2) Show that the convex, centrally symmetric set $X = \{(z_\tau) \in K_{\mathbf{R}} : \sum_{\tau} |z_\tau| < t\}$ has volume $\text{vol}(X) = 2^r \pi^s \frac{t^n}{n!}$. You do not need to turn in anything for this question, but just read the proof in III.2.15.
5. (Neukirch, section 5, exercise 3) Show that in every ideal $\mathfrak{a} \neq 0$ of $\mathcal{O}_K$ there exists an $a \neq 0$ such that $|N_{K/\mathbf{Q}}(a)| \leq M(\mathcal{O}_K : \mathfrak{a})$ where $M = \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|d_K|}$. (See the hint in the text.)
6. (Neukirch, section 6, exercise 3) Show that in every ideal class of an algebraic number field K of degree n , there exists an integral ideal $\mathfrak{a}$ such that $N(\mathfrak{a}) \leq M$, where M is the constant of the previous exercise.
7. (Neukirch, section 6, exercise 4) Show that the absolute value of the discriminant $|d_K|$ is > 1 for every algebraic number field $K \neq \mathbf{Q}$.