Mathematics 204A Homework 6
Due: Friday 12 November 2021

1. (Neukirch, section 9, exercise 1) Prove that if $L|K$ is a Galois extension of algebraic number fields with noncyclic Galois group, then there are at most finitely many nonsplit prime ideals of K.

2. (Neukirch, section 9, exercise 4) Let $L|K$ be a finite (not necessarily Galois) extension of algebraic number fields, and $N|K$ the normal closure of $L|K$. Show that a prime ideal $\mathfrak{p}$ of K is totally split in L if and only if it is totally split in N. (See the hint in the text.)

3. Suppose A is a Dedekind domain with field of fractions K, L is a finite separable extension of K, and B is the integral closure of A in L. Let I_A and I_B denote the ideal groups of A and B, i.e., the group of nonzero fractional ideals of A, respectively, B. Define $N : I_B \to I_A$ to be the unique group homomorphism defined as $N(\mathfrak{q}) = \mathfrak{p}^{[B/\mathfrak{q}:A/\mathfrak{p}]}$, where $\mathfrak{q}$ in B is prime and $\mathfrak{p} = A \cap \mathfrak{q}$. Prove that, if $\mathfrak{b} \subseteq B$ is an ideal, then $N(\mathfrak{b})$ is the ideal of A generated by the set $\{N_{L/K}(b) : b \in \mathfrak{b}\}$ of norms of elements of $\mathfrak{b}$. In particular, if $\mathfrak{b}$ is principal, then so is $N(\mathfrak{b})$.

   For this problem, either prove the above claim, or find a reference and read it. All that you need to turn in is a citation to the proof of the above claim.

4. Let ζ_{23} denote a primitive 23^{rd} root of unit, and set $L = \mathbf{Q}(\zeta_{23})$. This problem will show that the class group of L is nontrivial.

   (a) Observe that $K = \mathbf{Q}(\sqrt{-23})$ is the unique quadratic subfield of L.

   (b) Prove that $2\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p}$ nonprincipal but $\mathfrak{p}^3$ principal.

   (c) Suppose $\mathfrak{P}$ sits above $\mathfrak{p}$. Deduce that $\mathfrak{P}$ is nonprincipal, as follows. If $\mathfrak{P}$ were principal, then $N(\mathfrak{P}) = \mathfrak{p}^f$ would be principal (by Problem 3). But f divides 11.