Exam 1: Math 100C, Spring 2024

You have 50 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT,
(unless a problem specifies otherwise)

Name ______________________________

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ______________________________

There are 5 regular problems, worth 20 points each, and one bonus problem, worth 10 points.

Good luck!

Problem 1. *(20 points) For each example below of a field F and a polynomial $f(x) \in F[x]$, determine whether the polynomial $f(x)$ is irreducible in $F[x]$. You do not need to justify your answer for this question.*

1. (5 points) Let $F = \mathbf{Q}$ and $f(x) = x^3 - 5$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

   Irreducible. (E.g., by Eisenstein)

2. (5 points) Let $F = \mathbf{Q}(\sqrt{2})$ and $f(x) = x^2 - 2$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

   Reducible. This polynomial factors.

3. (5 points) Let $F = \mathbf{Q}(\sqrt{2})$ and $f(x) = x^3 - 3$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

   Irreducible. The extension $\mathbf{Q}(2^{1/2}, 3^{1/3})$ has degree three over $\mathbf{Q}(2^{1/2})$, because the degrees 2 (of $2^{1/2}$) and 3 (of $3^{1/3}$) are relatively prime.

4. (5 points) Let $F = \mathbf{R}$ and $f(x) = x^2 + 2x - 5$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

   Reducible. This quadratic has a real root by the quadratic formula.

Problem 2. *(20 points) Let* $i = \sqrt{-1} \in \mathbf{C}$ *and let* $\beta = 3^{1/4} \in \mathbf{R}$ *denote the positive fourth root of* 3*. We consider* i *and* β *as elements of the complex numbers* $\mathbf{C}$*.*

1. *(10 points) What is the degree of* $\mathbf{Q}(i, \beta)$ *over* $\mathbf{Q}(\beta)$*, i.e., what is* $[\mathbf{Q}(i, \beta) : \mathbf{Q}(\beta)]$*?*
   The degree is 2*, because* $x^2 + 1$ *is irreducible over* $\mathbf{R}$ *and* $\mathbf{Q}(\beta) \subseteq \mathbf{R}$*.*

2. *(10 points) What is the degree of* $\mathbf{Q}(i, \beta)$ *over* $\mathbf{Q}$*, i.e., what is* $[\mathbf{Q}(i, \beta) : \mathbf{Q}]$*?*
   The degree is 8*. By Eisenstein* $x^4 - 3$ *is irreducible in* $\mathbf{Q}[x]$*, so* $\mathbf{Q}(\beta)$ *has degree four over* $\mathbf{Q}$*. Now one applies part (1) and the multiplicativity of the degree.*

Problem 3. *(20 points) (This problem uses the same notation as the previous problem, but otherwise has no relation to it.) Let* $i = \sqrt{-1} \in \mathbf{C}$ *and let* $\beta = 3^{1/4} \in \mathbf{R}$ *denote the positive fourth root of* 3*. We consider* i *and* β *as elements of the complex numbers* $\mathbf{C}$*. Let* $\alpha = i\beta$ *and let* $L = \mathbf{Q}(\alpha)$ *be the subfield of* $\mathbf{C}$ *generated by* α*. Note that* $i = \sqrt{-1}$ *is being multiplied by* $\beta = 3^{1/4}$*, so* L *is a strict subfield of* $\mathbf{Q}(i, \beta)$*. Does there exist* $x, y, z \in L$ *so that* $2x^2 + y^2 + z^2 = -2$*? Be sure to justify your answer.*

The complex numbers α and β are both roots of the irreducible polynomial $x^4 - 3 \in \mathbf{Q}[x]$ (irreducible by Eisenstein), thus there is a field isomorphism $\sigma L \to \mathbf{Q}(\beta)$ taking α to β. As $\mathbf{Q}(\beta) \subseteq \mathbf{R}$, the equation $2x^2 + y^2 + z^2 = -2$ has no solutions in $\mathbf{Q}(\beta)$, and therefore there are also no solutions in L.

Problem 4. *(20 points) Recall that if F is a field, and V, W are finite dimensional vector spaces over F, then there is a unique isomorphism $\varphi : W \otimes V^* \to Hom(V, W)$ with the property that if $w \in W$ and $\ell \in V^*$, then $\varphi(w \otimes \ell)(v) = \ell(v)w$ for all $v \in V$. Suppose $V = F^3$, $W = V = F^3$, and b_1, b_2, b_3 is a basis of V. Let b_1', b_2', b_3' denote the basis of V^* dual to b_1, b_2, b_3. Set $S \in V \otimes V^*$ the element*

$$S = (b_1 + 2b_2 + 3b_3) \otimes b_1' + (-b_1 + b_2 - 3b_3) \otimes b_2' + (b_2 - 2b_3) \otimes b_3'.$$

As the element $\varphi(S) \in Hom(V, V)$ is a linear map from V to itself, it has a trace. What is this trace?

From the definition, one has $\varphi(S)(b_1) = b_1 + 2b_2 + 3b_3$, $\varphi(S)(b_2) = -b_1 + b_2 - 3b_3$, and $\varphi(S)(b_3) = b_2 - 2b_3$. Thus the trace is $1 + 1 + (-2) = 0$.

Problem 5. *(20 points) Suppose F is a field, V is a finite-dimensional F vector space, and*

$$T : V \to V, \quad S : V \to V$$

are linear maps. Suppose that the composite $S \circ T : V \to V$ is an isomorphism.

1. *(5 points) Explain why T must be injective.*
   Indeed, $T(v) = 0$ implies $ST(v) = 0$ implies $v = 0$ because ST is an isomorphism.

2. *(5 points) Explain why S must be surjective.*
   Indeed, if $w \in W$ then because ST is an isomorphism, there exists $v \in V$ with $ST(v) = w$. Hence S applied to $T(v)$ is w so S is surjective.

3. *(10 points) Explain why both S and T must be isomorphisms. (You may use the previous two parts for this part, even if you did not solve the previous two parts.)*
   A linear map on a finite-dimensional vector space is an isomorphism if and only if it is injective if and only if it is surjective (we proved this in class, as a consequence of the Rank-Nullity theorem). Thus part 3 follows from parts 1,2.

Problem 6. ***Bonus*** *(10 points) Let F be a field of characteristic 0 and V a finite-dimensional F vector space. Recall that a linear map $N : V \to V$ is said to be* nilpotent *if there exists a positive integer k so that $N^k = 0$ as a linear map on V. If N is nilpotent and $N^k = 0$, let*

$$\exp(N) = 1 + N + \frac{N^2}{2!} + \cdots + \frac{N^{k-1}}{(k-1)!} = \sum_{j=0}^{k-1} \frac{N^j}{j!}.$$

Prove that $\exp(N)$ *is an invertible linear transformation of V.*

The point is that one can give an explicit inverse: $\exp(-N) = 1 - N + N^2/2! + \cdots + (-1)^{k-1}\frac{N^{k-1}}{(k-1)!}$. One only has to explain that $\exp(N)\exp(-N) = 1 = \exp(-N)\exp(N)$. But if x is the variable in the power series ring $\mathbf{Q}[[x]]$, there is an identify of power series $\exp(x)\exp(-x) = 1$. This identity is simply a fact about binomial coefficients. Thus it applies equally well to prove $\exp(N)\exp(-N) = 1$, where one notes that both sides just have finite powers of N because N is nilpotent.

Alternatively (or what more or less amounts to the same thing), one could multiply out

$$\left(1 + N + \frac{N^2}{2!} + \cdots + \frac{N^{k-1}}{(k-1)!}\right)\left(1 - N + N^2/2! + \cdots + (-1)^{k-1}\frac{N^{k-1}}{(k-1)!}\right)$$

and get 1 by a simple fact about binomial coefficients.

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