Exam 1: Math 100C, Spring 2024

You have 50 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT,
(unless a problem specifies otherwise)

Name ____

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ____

There are 5 regular problems, worth 20 points each, and one bonus problem, worth 10 points.

Good luck!

Problem 1. *(20 points) For each example below of a field F and a polynomial $f(x) \in F[x]$, determine whether the polynomial $f(x)$ is irreducible in $F[x]$. You do not need to justify your answer for this question.*

1. (5 points) Let $F = \mathbf{Q}$ and $f(x) = x^3 - 5$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

2. (5 points) Let $F = \mathbf{Q}(\sqrt{2})$ and $f(x) = x^2 - 2$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

3. (5 points) Let $F = \mathbf{Q}(\sqrt{2})$ and $f(x) = x^3 - 3$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

4. (5 points) Let $F = \mathbf{R}$ and $f(x) = x^2 + 2x - 5$.

   ○ $f(x) \in F[x]$ is **irreducible**.
   ○ $f(x) \in F[x]$ is **reducible**.

Problem 2. *(20 points) Let* $i = \sqrt{-1} \in \mathbf{C}$ *and let* $\beta = 3^{1/4} \in \mathbf{R}$ *denote the positive fourth root of* 3*. We consider* i *and* β *as elements of the complex numbers* $\mathbf{C}$*.*

1. *(10 points) What is the degree of* $\mathbf{Q}(i, \beta)$ *over* $\mathbf{Q}(\beta)$*, i.e., what is* $[\mathbf{Q}(i, \beta) : \mathbf{Q}(\beta)]$*?*

2. *(10 points) What is the degree of* $\mathbf{Q}(i, \beta)$ *over* $\mathbf{Q}$*, i.e., what is* $[\mathbf{Q}(i, \beta) : \mathbf{Q}]$*?*

Problem 3. *(20 points) (This problem uses the same notation as the previous problem, but otherwise has no relation to it.) Let* $i = \sqrt{-1} \in \mathbf{C}$ *and let* $\beta = 3^{1/4} \in \mathbf{R}$ *denote the positive fourth root of* 3*. We consider* i *and* β *as elements of the complex numbers* $\mathbf{C}$*. Let* $\alpha = i\beta$ *and let* $L = \mathbf{Q}(\alpha)$ *be the subfield of* $\mathbf{C}$ *generated by* α*. Note that* $i = \sqrt{-1}$ *is being multiplied by* $\beta = 3^{1/4}$*, so* L *is a strict subfield of* $\mathbf{Q}(i, \beta)$*. Does there exist* $x, y, z \in L$ *so that* $2x^2 + y^2 + z^2 = -2$*? Be sure to justify your answer.*

Problem 4. *(20 points) Recall that if F is a field, and V, W are finite dimensional vector spaces over F, then there is a unique isomorphism $\varphi : W \otimes V^* \to Hom(V, W)$ with the property that if $w \in W$ and $\ell \in V^*$, then $\varphi(w \otimes \ell)(v) = \ell(v)w$ for all $v \in V$. Suppose $V = F^3$, $W = V = F^3$, and b_1, b_2, b_3 is a basis of V. Let b_1', b_2', b_3' denote the basis of V^* dual to b_1, b_2, b_3. Set $S \in V \otimes V^*$ the element*

$$S = (b_1 + 2b_2 + 3b_3) \otimes b_1' + (-b_1 + b_2 - 3b_3) \otimes b_2' + (b_2 - 2b_3) \otimes b_3'.$$

As the element $\varphi(S) \in Hom(V, V)$ is a linear map from V to itself, it has a trace. What is this trace?

Problem 5. *(20 points) Suppose F is a field, V is a finite-dimensional F vector space, and*

$$T : V \to V, \quad S : V \to V$$

are linear maps. Suppose that the composite $S \circ T : V \to V$ is an isomorphism.

1. *(5 points) Explain why T must be injective.*

2. *(5 points) Explain why S must be surjective.*

3. *(10 points) Explain why both S and T must be isomorphisms. (You may use the previous two parts for this part, even if you did not solve the previous two parts.)*

Problem 6. ***Bonus*** *(10 points) Let F be a field of characteristic 0 and V a finite-dimensional F vector space. Recall that a linear map $N : V \to V$ is said to be* nilpotent *if there exists a positive integer k so that $N^k = 0$ as a linear map on V. If N is nilpotent and $N^k = 0$, let*

$$\exp(N) = 1 + N + \frac{N^2}{2!} + \cdots + \frac{N^{k-1}}{(k-1)!} = \sum_{j=0}^{k-1} \frac{N^j}{j!}.$$

Prove that $\exp(N)$ *is an invertible linear transformation of V.*

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