Mathematics 100C Homework 9
Due: Friday June 7 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TA if you have any questions.

1. Suppose A is diagonalizable on a finite-dimensional vector space V over a field F, $U \subseteq V$ is a subspace, and A takes U to U. Prove that A is diagonalizable on U. **Hint**: By definition of A being diagonalizable, there exists $\lambda_1, \ldots, \lambda_r \in F$, distinct, and subspaces $V_j \subseteq V$ so that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ and $Av_j = \lambda_j v_j$ for all $v_j \in V_j$. Now suppose $u \in U$, $u = v_1 + \cdots + v_r$ with $v_j \in V_j$. We claim that $v_j \in U$ for all j. To see this for $j = r$, consider $(A - \lambda_1) \cdots (A - \lambda_{r-1})u$, which must be in U by assumption. Reasoning like this, one obtains $U = (U \cap V_1) \oplus \cdots (U \cap V_r)$. Now deduce that A is diagonalizable on U.

2. Suppose $A_1, \ldots, A_n$ are diagonalizable operators on a finite-dimensional vector space V over a field F, and suppose moreover that A_i commutes with A_j for all i, j. Prove that the A_j are *simultaneously diagonalizable* on V, i.e., there exists a basis $b_1, \ldots, b_N$ of V so that b_j is an eigenvector of A_k for all j, k. **Hint**: Let $A = A_n$, and write $V = V_1 \oplus \cdots \oplus V_r$ as in the previous problem, so that $A_n v_j = \lambda_j v_j$ for all $v_j \in V_j$ and the λ_j are distinct. Verify that $A_1, \ldots, A_{n-1}$ takes V_j to V_j for all j. By the previous problem, the restriction of $A_1, \ldots, A_{n-1}$ to V_j is diagonalizable. Now induct on n to see that the restriction of $A_1, \ldots, A_{n-1}$ to V_j are simultaneously diagonalizable.

3. Suppose G is a finite **abelian** group, and $\rho : G \to \mathrm{GL}(V)$ is a complex irreducible representation. Prove directly that $\dim(V) = 1$. (We will prove this in class in a somewhat indirect way. I am asking you to give a different proof, using the previous problem.) **Hint**. Let $G = \{g_1, \ldots, g_n\}$ and let $A_j = \rho(g_j)$. The A_j are diagonalizable (we proved this), and they all commute because G is abelian. Now use the previous problem.