

# Mathematics 100C Homework 8

## Due: Friday May 31 2024

**Instructions:** Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TA if you have any questions.

1. Let  $D_n$  denote the dihedral group of order  $2n$ , defined by generators and relations as

$$D_n = \langle \sigma, \tau : \sigma^n = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle.$$

Define a *complex* representation of  $D_n$  on  $\mathbf{C}^2$  as  $\rho : D_n \rightarrow \mathrm{GL}_2(\mathbf{C})$  given by  $\rho(\sigma) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$  and  $\rho(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Prove that if  $n \geq 3$  then  $\rho$  is irreducible. **Hint:** If  $\rho$  were reducible, there would have to be a vector  $v \in \mathbf{C}^2$  which is an eigenvector for every  $\rho(g)$ ,  $g \in D_n$ .

2. Suppose  $G$  is a finite group, and  $V$  is a finite-dimensional  $\mathbf{C}$  vector space, and  $\rho : G \rightarrow \mathrm{GL}(V)$  is a representation. For  $v \in V$ , define

$$\mathbf{C}[G](v) = \mathrm{Span}_{g \in G} \{g \cdot v\} = \left\{ \sum_{g \in G} a_g \rho(g)(v) : a_g \in \mathbf{C} \right\},$$

the span of all the elements  $\rho(g)(v)$  as  $g$  ranges over elements of  $G$ . Prove that  $V$  is irreducible if and only if  $\mathbf{C}[G](v) = V$  for all  $v \neq 0$  in  $V$ .

3. The standard representation of the symmetric group  $S_n$  is defined as follows: Let  $S_n$  act on  $\mathbf{C}^n$  by permutations,  $\sigma(v_1, \dots, v_n) = (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)})$ . Set  $W \subseteq \mathbf{C}^n$  to be the subspace of elements whose coordinates sum to 0,  $W = \{v = (v_1, \dots, v_n) : v_1 + \dots + v_n = 0\}$ , so that  $W$  is  $n - 1$ -dimensional. We let  $S_n$  act on  $W$  by the restriction of the  $S_n$  action on  $\mathbf{C}^n$ . The representation of  $S_n$  on  $W$  is called the standard representation. Prove that it is irreducible. **Hint:** We will apply the criterion of the previous problem. Suppose  $w = (w_1, \dots, w_n) \in W$  is not zero. Then there exists  $i, j \in \{1, 2, \dots, n\}$  with  $i \neq j$  so that  $w_i \neq w_j$ . Let  $\tau$  be the permutation of  $S_n$  that exchanges  $i$  with  $j$  and leaves the other indices fixed. Then  $\tau \cdot w - w = \alpha(e_i - e_j)$  for some  $\alpha \neq 0$ . Here  $e_k$  is the  $k^{\mathrm{th}}$  standard basis vector of  $\mathbf{C}^n$ . Now check that the span of the  $S_n$  translates of  $e_i - e_j$  is all of  $W$ .