Mathematics 100C Homework 7
Due: Friday May 24 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TA if you have any questions.

1. Suppose F is a field of characteristic 0, and K/F is a finite Galois extension. Suppose K_1 and K_2 are subfields of K containing F, and let $H_1 = Gal(K/K_1)$, $H_2 = Gal(K/K_2)$. Set $L = K_1K_2$ the smallest subfield of K containing both K_1 and K_2. Prove that $Gal(K/L) = H_1 \cap H_2$.

2. Suppose F is a field of characteristic 0, and K/F is a Galois extension with Galois group isomorphic to the symmetric group S_n. The purpose of this problem is to prove that K is the splitting field over F of an irreducible polynomial of degree n.

   (a) Let $H_1 \leq Gal(K/F)$ be the subgroup of $Gal(K/F)$ corresponding to the permutations that fix the element (1) of $\{1, 2, \ldots, n\}$, so that $H_1 \simeq S_{n-1}$. Let $L_1 = K^{H_1}$ be the fixed field of H_1. Let ϵ_1 be a primitive element for L_1 over F, and let $f(x) \in F[x]$ be the monic irreducible polynomial for ϵ_1. Prove that $f(x) \in F[x]$ is irreducible of degree n.

   (b) Let $\epsilon_2, \ldots, \epsilon_n$ be the other roots of $f(x)$ in K, set $L_j = F(\epsilon_j)$, and let $H_j = Gal(K/L_j)$. Observe that, for every $\sigma \in Gal(K/F)$, $\sigma(\epsilon_1) = \epsilon_j$ for some j, and $Gal(K/L_j) = \sigma H_1 \sigma^{-1}$ if $\sigma(\epsilon_1) = \epsilon_j$. Prove that, for each $k \in \{1, 2, \ldots, n\}$, there exists $\sigma_k \in Gal(K/F)$ with $\sigma_k(\epsilon_1) = \epsilon_k$. **Hint**: One way to do this is to observe that H_1 has exactly n conjugates in $Gal(K/F)$.

   (c) Prove that the composite field $L_1 \cdots L_n = F(\epsilon_1, \ldots, \epsilon_n)$ is equal to K. **Hint**: Use the previous problem, and the group theory of S_n.

3. Suppose $a, b \in \mathbf{Q}^\times$, and $a \notin (\mathbf{Q}^\times)^2$, $b \notin (\mathbf{Q}^\times)^2$, and $ab \notin (\mathbf{Q}^\times)^2$. Let $K = \mathbf{Q}(\sqrt{a}, \sqrt{b})$. Prove that $K/\mathbf{Q}$ is Galois with Galois group isomorphic to the Klein 4 group.