Mathematics 100C Homework 4
Due: Friday May 3 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TA if you have any questions.

1. (Chapter 15, problem 6.1) Let F be a field of characteristic 0, and let f' denote the derivative of f in $F[x]$. Suppose g is an irreducible polynomial that is a common divisor of f and f'. Prove that g^2 divides f.

2. (Chapter 15, problem 6.2) Let F be a field of characteristic not 2. Recall that we proved in class that if K is a quadratic extension of F, there exists $\delta \in K \setminus F$ so that $\delta^2 \in F^\times$.

   (a) Let $K = F(\delta)$ be a quadratic extension of F, and suppose $\mu \in K \setminus F$ satisfies $\mu^2 \in F$. Prove that $\mu = c\delta$ for some $c \in F^\times$.

   (b) Given a quadratic extension K of F, take an arbitrary $\delta \in K \setminus F$ that satisfies $\delta^2 \in F^\times$, and define $\Delta(K)$ to be the class of $\delta^2 \in F^\times/(F^\times)^2$. Observe that by the previous part, $\Delta(K)$ is well-defined, i.e., independent of the choice of δ, and that $\Delta(K)$ is not the identity class in this quotient group. Suppose K' is another quadratic extension of F, and $\Delta(K) = \Delta(K')$ in $F^\times/(F^\times)^2$. Prove that K is isomorphic to K'.

   (c) Conversely, suppose $a \in F^\times$ is not in $(F^\times)^2$. Prove that there is a quadratic extension K of F with $\Delta(K)$ being the class of a in $F^\times/(F^\times)^2$.

3. Suppose K is a field of size of $q = p^r$. Define $\varphi : K \to K$ as $\varphi(\alpha) = \alpha^p$. Prove that φ is a field isomorphism.

4. (Chapter 15, problem 7.10) Let F be a finite field, and $f(x) \in F[x]$ a nonconstant polynomial whose derivative is the zero polynomial. Prove that f is reducible in $F[x]$.