Mathematics 100C Homework 3
Due: Friday 26 April 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TA if you have any questions.

1. (Chapter 15, problem 2.3) Let $\beta = \omega 2^{1/3}$, where $\omega = e^{2\pi i/3}$. Set $K = \mathbf{Q}(\beta)$. Fix a positive integer k. Prove that the equation $x_1^2 + x_2^2 + \cdots + x_k^2 = -1$ has no solution with all $x_i \in K$.

2. (Discussion section) Let F be a field, and $K = F(\alpha)$ a field extension of F with $[K : F] = n$ with n odd. Prove that α^2 also generates K over F, i.e., that $K = F(\alpha^2)$.

3. (Chapter 15, problem 3.3) Prove that the polynomial $x^4 + 3x + 3$ is irreducible over the field $\mathbf{Q}[2^{1/3}]$.

4. Suppose $F \subseteq K \subseteq L$ are field extensions. Assume that K is algebraic over F, which means that every element of K is algebraic over F. Now let $\alpha \in L$ be algebraic over K. Prove that α is algebraic over F.

5. (Chapter 15, problem 3.8) Suppose α and β are complex numbers for which $\alpha + \beta$ and $\alpha\beta$ are algebraic. Prove that α and β are individually algebraic.

6. (Chapter 15, problem 3.9) Suppose $f(x) \in \mathbf{Q}[x]$ and $g(x) \in \mathbf{Q}[x]$ are irreducible. Let $\alpha, \beta \in \mathbf{C}$ be such that $f(\alpha) = 0$ and $g(\beta) = 0$. Set $K = \mathbf{Q}(\alpha)$ and $L = \mathbf{Q}(\beta)$. Prove that $f(x)$ is irreducible in $L[x]$ if and only if $g(x)$ is irreducible in $K[x]$.