Mathematics 100C Homework 2
Due: Friday 19 April 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TA if you have any questions.

1. Suppose F is a field, and $X, Y \in M_n(F)$. Recall that the trace of a matrix $M = (m_{ij}) \in M_n(F)$ is defined to be the sum of the diagonal entries of M: $\mathrm{tr}(M) = \sum_{i=1}^{n} m_{ii}$.
   (a) Prove that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$.
   (b) Now suppose V is an n-dimensional F vector space, and $T : V \to V$ is a linear map. Let B be a basis of V and let M be the matrix of T with respect to B. Define $\mathrm{tr}(T) = \mathrm{tr}(M)$, the trace of the associated matrix. Prove that this is independent of the choice of basis, and thus well-defined. **Hint**: How does the matrix M change if you change basis?

2. (Important fact about tensor products) Suppose V, W are F vector spaces of dimensions n, m respectively. Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W.

   Theorem *With notation as above, the mn elements $v_i \otimes w_j$ form a basis of $V \otimes W$.*

   Read and work through the exercises below which prove this theorem. You do not need to turn in anything for this question or the questions below that prove this.

3. Suppose $\dim(V) = n$ and $\dim(W) = m$, both finite. Define a map $\varphi : W \otimes V^* \to Hom_F(V, W)$ as $\varphi(\sum_i w_i \otimes \ell_i)(v) = \sum_i \ell_i(v) w_i$.
   (a) Prove that φ is well-defined. **Hint**: Use the Universal Mapping Property of tensor products below.
   (b) Let $B = (b_1, \ldots, b_n)$ be a basis of V and $(b'_1, \ldots, b'_n)$ the dual basis of V^*. Suppose $T \in Hom_F(V, W)$. Let $T_1 \in W \otimes V^*$ be $T_1 = \sum_{1 \le i \le n} T(b_i) \otimes b'_i$. Prove that $\varphi(T_1) = T$.
   (c) Deduce that φ is an isomorphism. **Hint** φ is surjective and the dimensions match.

Facts about tensor products for Question 2 above. *You do not need to turn in solutions to any of the problems below.*

1. (Universal property of tensor products) Recall that if V, W are F vector spaces, then in Homework 1 we defined the F vector space $V \otimes W$. In this problem, we specify the Universal Mapping Property of the tensor product. To set this up, suppose U is an F vector space. A **bilinear map** from $V \times W$ to U is a map $L : V \times W \to U$ satisfying
   (a) $L(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 L(v_1, w) + \alpha_2 L(v_2, w)$ for all $\alpha_1, \alpha_2 \in F$ and all $v_1, v_2 \in V$, $w \in W$;
   (b) $L(v, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 L(v, w_1) + \alpha_2 L(v, w_2)$ for all $\alpha_1, \alpha_2, \in F$ and all $w_1, w_2 \in W$, $v \in V$.

The tensor product $V \otimes W$ comes equipped with a bilinear map $T : V \times W \to V \otimes W$ given by $T(v, w) = v \otimes w$. Prove that $V \otimes W$ has the following **Universal Mapping Property**: Suppose $L : V \times W \to U$ is a bilinear map. Then there exists a unique linear map $\varphi : V \otimes W \to U$ so that $L = \varphi \circ T$.

2. Suppose $V = F$. Define a bilinear map $L : F \times W \to W$ as $L(\lambda, w) = \lambda w$. Prove that L induces an isomorphism $F \otimes W \simeq W$. **Hint**: The universal mapping property implies there exists a unique $\varphi : F \otimes W \to W$ satisfying $L = \varphi \circ T$. We aim to prove that φ is an isomorphism. To do this, show that $L : F \times W \to W$ satisfies the same universal mapping property that T does. Then there exists a unique $\Psi : W \to F \otimes W$ so that $\Psi \circ L = T$. Now, by the uniqueness part of the Universal Mapping Property, conclude that $\varphi \circ \Psi$ and $\Psi \circ \varphi$ are the identity on their respective domains.

3. Suppose $V = V_1 \oplus V_2$. Prove that $V \otimes W \simeq V_1 \otimes W \oplus V_2 \otimes W$. **Hint**: Show that $V_1 \otimes W \oplus V_2 \otimes W$ satisfies the same universal mapping property as $V \otimes W$. Deduce that if $\dim_F(V) = n$ and $\dim_F(W) = m$, then $\dim_F(V \otimes W) = mn$.

4. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Prove that the mn elements $v_i \otimes w_j$ form a basis of $V \otimes W$. **Hint** In homework 1, you proved that these elements span $V \otimes W$. But $\dim_F(V \otimes W) = mn$, so they are a basis.