Mathematics 100C Homework 1
Due: Friday 12 April 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TA if you have any questions.

1. (Dual vector space) Suppose V is a vector space over a field F. Let $V^* = \{\ell : V \to F, \ell \text{ linear}\}$ denote the set of linear maps from V to F. I.e., if $\ell : V \to F$, then $\ell \in V^*$ if and only if $\ell(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \ell(v_1) + \alpha_2 \ell(v_2)$ for all $\alpha_1, \alpha_2 \in F$ and all $v_1, v_2 \in V$. The set V^* is called the dual vector space of V. Define an addition and scalar multiplication on V^* as $(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v)$ if $\ell_1, \ell_2 \in V^*$ and $v \in V$ and $(\alpha\ell)(v) = \alpha\ell(v)$ if $\alpha \in F$, $\ell \in V^*$ and $v \in V$. Verify for yourself that these rules make V^* into an F vector space. **Notation**: If $\ell \in V^*$ and $v \in V$, it is sometimes convenient to define $\langle \ell, v \rangle := \ell(v)$. You do not need to turn in anything for this problem.

2. (Dual bases) Suppose V is a finite-dimensional F vector space, and let V^* denote the dual vector space. (See problem 1). Let $b_1, \ldots, b_n$ be a basis of V. A list $\ell_1, \ell_2, \ldots, \ell_n$ in V^* is called the dual basis of $(b_1, \ldots, b_n)$ if $\langle \ell_i, b_j \rangle = \delta_{ij}$ (Kronecker delta). I.e., $\ell_i(b_i) = 1$ and $\ell_i(b_j) = 0$ if $i \neq j$. Prove that, if the basis $(b_1, \ldots, b_n)$ is given, then there exists a unique dual basis $(\ell_1, \ldots, \ell_n)$ to $(b_1, \ldots, b_n)$. Moreover, prove that $(\ell_1, \ldots, \ell_n)$ form a basis of V^*.

3. (Quotient vector spaces) Suppose F is a field, V an F vector space, and $U \subseteq V$ a subspace. Let V/U denote the quotient of V by U in the sense of abelian groups. Define a scalar multiplication on V/U by $\lambda(v + U) = \lambda v + U$. Here $\lambda \in F$ and $v \in V$. Prove that this scalar multiplication is well-defined (i.e., independent of the choice of coset representative v), and makes V/U into an F vector space. You do not need to turn in anything for this problem.

4. (Free vector spaces) Suppose S is a set and F is a field. The free vector space $\mathcal{F}$ on S is the set whose elements are finite sums
$$\lambda_{s_1}[s_1] + \lambda_{s_2}[s_2] + \cdots \lambda_{s_n}[s_n].$$
Here, for each $s \in S$ there is an associated element $[s] \in \mathcal{F}$, and the λ_{s_j} are elements of F. We write such an expression as $\sum_{s \in S} \lambda_s[s]$ with the understanding that the $\lambda_s = 0$ for all but finitely many s. We define an addition on $\mathcal{F}$ by adding two such expressions componentwise:
$$\sum_{s \in S} \lambda_s[s] + \sum_{s \in S} \mu_s[s] = \sum_{s \in S} (\lambda_s + \mu_s)[s].$$
Similarly, we define a scalar multiplication as $\lambda \left(\sum_{s \in S} \lambda_s[s]\right) = \sum_{s \in S} \lambda\lambda_s[s]$. Verify for yourself that these operations make $\mathcal{F}$ into an F-vector space. You do not need to turn in anything for this problem.

5. (Recollection of tensor product) Suppose F is a field, and V, W are F vector spaces. Let $S = V \times W$, and let $\mathcal{F}$ be the free vector space on the set S. Define $\mathcal{G} \subseteq \mathcal{F}$ as the subspace spanned by the following elements:

(a) $[(m_1 + m_2, n)] - [(m_1, n)] - [(m_2, n)]$ for all $m_1, m_2 \in V$ and $n \in W$;

(b) $[(m, n_1 + n_2)] - [(m, n_1)] - [(m, n_2)]$ for all $m \in V$ and $n_1, n_2 \in W$;

(c) $\lambda[(m, n)] - [(\lambda m, n)]$ for all $\lambda \in F$, $m \in V$ and $n \in W$;

(d) $\lambda[(m, n)] - [(m, \lambda n)]$ for all $\lambda \in F$, $m \in V$ and $n \in W$;

Let $V \otimes W$ denote the quotient vector space $\mathcal{F}/\mathcal{G}$. The vector space $V \otimes W$ is called the tensor product of V and W. If $v \in V$ and $w \in W$, we write $v \otimes w$ for the image of the element $[(v, w)]$ in $V \otimes W$. Observe that the symbol $\otimes$ has the following properties:

(a) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$; $v_1, v_2 \in V$, $w \in W$;

(b) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$; $v \in V$, $w_1, w_2 \in W$.

(c) $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$; $\lambda \in F$, $v \in V$, $w \in W$.

Suppose $v_1, \ldots, v_n$ span V, and $w_1, \ldots, w_m$ span W. Prove the the $m \cdot n$ elements $v_i \otimes w_j$, $1 \leq i \leq n$, $1 \leq j \leq m$ span $V \otimes W$.