Final Exam Practice Problems

Instructions This worksheet is made to help you review for the final exam. The first four "questions" are just a review of some of the material we covered this term. I would read through these questions carefully, and make sure you understand everything written. The final few questions are just to give you some extra practice with representation theory.

The exam will consist of 10 questions. The exam will cover material from the entire course. You can expect exam questions to be similar to homework questions, discussion questions, and the questions on this worksheet. *Some exam questions may be exactly the same as a homework question, discussion section question, or a question from this worksheet.*

Problem 1. *Review for yourself the following facts about linear algebra:*

1. *Suppose V is a vector space over a field F, $L = (u_1, \ldots, u_r)$ is a linearly independent list of vectors of V, and $S = (w_1, \ldots, w_s)$ is a spanninng list of vectors of V. Then $r \leq s$. In particular, if V has a finite spanning set, then every basis of V has the same size, called the dimension of V.*
2. *Given a finite spanning set S of a vector space V, one can delete elements from S to obtain a basis of V. Given a linearly independent list L of vectors of V, one can add elements of S to L to obtain a basis of V. As a consequence, if $U \subseteq V$ is a subspace, then $\dim(U) \leq \dim(V)$ with equality if and only if $U = V$.*
3. *Suppose $T : V \to W$ is a linear map, and V is finite-dimensional. Then $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{Im}(T))$. This is called the Rank-Nullity theorem. If $\dim(V) = \dim(W) < \infty$, then T is injective if and only if T is surjective if and only if T is an isomorphism.*
4. *A linear map T from a finite-dimensional vector space V to itself has a trace, $\mathrm{tr}_V(T)$. The trace is conjugation invariant, i.e., $\mathrm{tr}(gTg^{-1}) = \mathrm{tr}(T)$ for all $g \in \mathrm{GL}(V)$.*
5. *If V and W are finite-dimensional vector spaces with bases $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$, then $V \otimes W$ is a finite-dimensional vector space, and it has a basis $\{e_i \otimes f_j\}$. In particular, $\dim(V \otimes W) = mn = \dim(V)\dim(W)$. Moreover, the map $\Phi : W \otimes V \to Hom(V, W)$ given by $\Phi(w \otimes \ell)(v) = \ell(v)w$ is an isomorphism.*

Problem 2. *Review for yourself the following facts about finite fields:*

1. *Finite fields always have prime characteristic. If F is a finite field of characteristic p, then the size of F is p^r for some positive integer r.*
2. *If F_1, F_2 are two finite fields of the same size, then F_1 is isomorphic to F_2. Conversely, if r is a postive integer and p is a prime number, then there exists a field F of size p^r.*
3. *The Frobenius map $\varphi : F \to F$ on a finite field is an isomorphism. If $|F| = p^r$, then $\mathrm{Aut}(F/\mathbf{F}_p)$ is cyclic of order r generated by Frobenious. In particular, if $q = p^r$, then $x^q = x$ for all $x \in F$.*
4. *The subfields of a finite field of order p^r are in bijection with the positive integer divisors of r (in line with the main theorem of Galois theory, even though we only developed Galois theory in the case of characteristic 0.)*

Problem 3. *Review for yourself the following facts about Galois theory. All fields in this question* *are characteristic* 0.

1. *If* K/F *is a finite extension,* $|\mathrm{Aut}(K/F)| \leq [K : F]$. *By definition,* K/F *is Galois if this inequality is an equality.*
2. *Suppose* F *is a field, and* L *is a finite extension field of* F. *Then there exists a finite extension* K *of* L *so that* K/F *is Galois.*
3. *A finite extension* K *of* F *is Galois if and only if* K *is a splitting field of some polynomial* $f(x) \in F[x]$.
4. *Suppose* K/F *is Galois with* $Gal(K/F) = G$. *Then there is an order-reversing bijection between intermeidiate fields and subgroups of* G. *This bijection sends the intermediate field* L *to* $Gal(K/L)$ *and the subgroup* H *to the fixed field* K^H. *In particular, one has* $K^G = F$.
5. *The Splitting theorem holds: If* K/F *is a splitting field, equivalently, a Galois extension, and* $f(x) \in F[x]$ *is an irreducible polynomial with at least one root in* K, *then* f *splits in* K.
6. *Suppose* K/F *is Galois, and* $\beta = \beta_1 \in K$. *Let* $\{\beta_1, \ldots, \beta_r\}$ *be the* $Gal(K/F)$ *orbit of* β_1. *Then* $g(x) := (x - \beta_1) \cdots (x - \beta_r) \in F[x]$, *and in fact is the irreducible polynomial for* $\beta = \beta_1$ *over* F.
7. *If* K/F *is a finite Galois extension, and* L *is an intermediate field, then* L *is Galois over* F *if and only if* $Gal(K/L)$ *is a normal subgroup of* $Gal(K/F)$. *In this case, the restriction map* $Gal(K/F) \to Gal(L/F)$ *is well-defined and surjective with kernel* $Gal(K/L)$.

Problem 4. *Please review for yourself the following facts about representation theory. In this question,* G *is a finite group, and all vector spaces are finite-dimensional over the complex numbers* $\mathbf{C}$.

1. *A representation of* G *is a group homomorphism* $\rho : G \to GL(V)$. *Equivalently, a representation is an action of* G *on* V *by linear automorphisms.*
2. *A representation* V *of* G *is irreducible if the only* G*-invariant subspaces of* V *are* $\{0\}$ *and* V. *Every representation can be written as a direct sum of irreducible representations.*
3. *The number of irreducible representations of* G *up to isomorphism is the number* r *of conjugacy classes in* G. *Let* $\rho_1, \ldots, \rho_r$ *be representatives for the isomorphism classes of the irreducible representations, with* $d_j = \dim(\rho_j)$. *Then* $|G| = d_1^2 + \cdots + d_r^2$.
4. *The character of a representation* $\rho : G \to \mathrm{GL}(V)$ *is defined as* $\chi_\rho(g) = \mathrm{tr}_V(\rho(g))$. *Thus,* χ_ρ *is a function from* G *to* $\mathbf{C}$. *The character is a class function, i.e., it is constant on conjugacy classes. If* $\varphi, \psi : G \to \mathbf{C}$ *are two class functions, we define a Hermitian pairing* $\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g)$.
5. *Schur's lemma says if* $\rho : G \to \mathrm{GL}(V)$ *and* $\rho' : G \to \mathrm{GL}(V')$ *are irreducible representations, then the space of* G*-invariant maps* $\mathrm{Hom}_G(V, V')$ *is* 0 *if* ρ *is not isomorphic to* ρ', *and is one-dimensional if* ρ *is isomorphic to* ρ'.
6. *If* ρ, ρ' *are representations (not-necessarily irreducible), then* $\langle \chi_{\rho'}, \chi_\rho \rangle$ *is a non-negative integer, and in fact* $\langle \chi_{\rho'}, \chi_\rho \rangle$ *is the dimension of the space of* G*-invariant linear maps* $\mathrm{Hom}_G(V, V')$. *In particular, if* ρ, ρ' *are irreducible, then* $\langle \chi_{\rho'}, \chi_\rho \rangle$ *is* 1 *or* 0 *depending on if* ρ *is isomorphic to* ρ' *or not.*

7. *Every irreducible representation of a finite abelian group is one-dimensional.*

Problem 5. *Let* $D_n = \langle \sigma, \tau : \sigma^n = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ *denote the dihedral group of order* $2n$*. Prove that every irreducible representation* $\rho : D_n \to \mathrm{GL}(V)$ *of* D_n *is either one or two-dimensional, as follows: Let* v *be an eigenvector for* σ*, and set* $v' = \tau \cdot v$*. Prove that* $\mathbf{C}v + \mathbf{C}v' \subseteq V$ *is* D_n*-invariant.*

Problem 6. *Let* $\rho_{perm} : S_n \to \mathbf{C}^n$ *denote the permutation representation. Let* $W \subseteq \mathbf{C}^n$ *be the subspace of vectors whose coordinates sum to* 0*. Recall that there is a representation* $\rho_{std} : S_n \to \mathrm{GL}(W)$ *coming from restricting the permutation action of* S_n *on* $\mathbf{C}^n$*.*

1. *Compute the character of* ρ_{std} *and* ρ_{perm} *in terms of the number of fixed points of a permutation* $\sigma \in S_n$*.*
2. *For* $\sigma \in S_n$*, let* $f(\sigma)$ *denote the number of fixed points of* σ*, i.e, the number of* $j \in \{1, 2, \ldots, n\}$ *for which* $\sigma(j) = j$*. Prove that* $\sum_{\sigma \in S_n} f(\sigma)^2 = 2 \cdot n!$*.*

Problem 7. *Suppose* $G = \mathbf{Z}/n_1\mathbf{Z} \times \cdots \times \mathbf{Z}/n_r\mathbf{Z}$ *is a finite abelian group. Write down explicitly every irreducible representation of* G*.*