Final Exam: Math 100C, Spring 2024

You have 180 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT,
(unless a problem specifies otherwise)

Name ____

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ____

There are 10 regular problems, worth 20 points each, and two bonus problems, worth 10 points each.

Good luck!

Problem 1. *(20 points) Let F be a field and V a finite-dimensional F vector space. Suppose that A and B are linear maps from V to V that commute. Recall that a linear map $T : V \to V$ is said to be* ***diagonalizable*** *if there is a basis $e_1, \dots, e_n$ of V and elements $\lambda_1, \dots, \lambda_n \in F$ so that $Te_j = \lambda_j e_j$ for all j. Suppose that A is diagonalizable on V with* ***distinct eignenvalues****. That is, there is a basis $b_1, \dots, b_n$ of V and $\alpha_1, \dots, \alpha_n \in F$ so that $Ab_j = \alpha_j b_j$ for all j, and $\alpha_i \neq \alpha_j$ if $i \neq j$. Prove that B is diagonalizable on V.*

Problem 2. *(20 points) Let F be a field, $n > 1$ a positive integer, and V an n-dimensional F vector space. Suppose $T : V \to V$ is a linear map that has the following property: There is a basis $e_1, \ldots, e_n$ of V for which $T(e_1) = e_n$ and $T(e_j) = e_{j-1}$ if $j = 2, 3, \ldots, n$. What is the trace of T?*

Problem 3. *(20 points) In class, we proved that if F is a finite field of size q, then $\alpha^q = \alpha$ for all $\alpha \in F$. Please reprove this fact.* ***Hint****: Apply Lagrange's theorem for the group $F^\times$.*

Problem 4. *(20 points) Recall that, for a prime number p, $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ denotes the field of size p. For this question, you do not need to justify your response.*

1. *(10 points) Find a polynomial $f(x) \in \mathbf{F}_2[x]$ so that $\mathbf{F}_2[x]/\langle f(x)\rangle$ is a field of size 8.*

2. *(10 points) Find a polynomial $g(x) \in \mathbf{F}_3[x]$ so that $\mathbf{F}_3[x]/\langle g(x)\rangle$ is a field of size 27.*

Problem 5. *(20 points) For a positive integer* n*, let* $\zeta_n = e^{2\pi i/n}$ *and* $K = \mathbf{Q}(\zeta_n)$*. Suppose* $\beta \in K^\times$*, and let* $L = K(\beta^{1/n})$*.*

1. *(10 points) Explain why the extension* L/K *is Galois.*

2. *(10 points) Prove that the Galois group* $Gal(L/K)$ *is cyclic.* ***Hint****: Let* μ_n *denote the set of* n^{th} *roots of unity in* $\mathbf{C}$*. Consider the map* $Gal(L/K) \to \mu_n$ *defined by* $\sigma \mapsto \frac{\sigma(\beta^{1/n})}{\beta^{1/n}}$*.*

Problem 6. *(20 points) Suppose F is a field of characteristic 0, and L is a finite extension of F. Must the extension L over F have finitely many intermediate fields? Be sure to justify your response.* ***Hint:*** *There exists a finite extension K of L so that K is Galois over F.*

Problem 7. *(20 points) For a positive integer n, define*

$$\alpha_n = (1+\sqrt{2}+\sqrt{3})^n + (1-\sqrt{2}+\sqrt{3})^n + (1+\sqrt{2}-\sqrt{3})^n + (1-\sqrt{2}-\sqrt{3})^n.$$

Is α_n a rational number for all positive integers n? If so, prove it. If not, explain why not.

Problem 8. *(20 points) For a positive integer n, let*

$$D_n = \langle \sigma, \tau : \sigma^n = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$$

denote the dihedral group of order $2n$. Prove that every irreducible representation of D_n has dimension 1 or 2. ***Hint****: Suppose $\rho : D_n \to \mathrm{GL}(V)$ is an irreducible representation. Let $v \in V$ be an eigenvector for $\rho(\sigma)$. Show that $\mathbf{C}v + \mathbf{C}(\tau \cdot v)$ is a D_n-invariant subspace of V.*

Problem 9. *(20 points) Suppose G is a finite group, and $\rho : G \to \mathrm{GL}(V)$ is a representation. Let $\chi_\rho(g) = \mathrm{tr}_V(\rho(g))$ denote the character of ρ. Suppose that $\sum_{g \in G} |\chi_\rho(g)|^2 = 2 \cdot |G|$. Prove that ρ is the direct sum of two distinct irreducible representations.* ***Note****: Here $|G|$ denotes the order of the group G, and, for a complex number $z \in \mathbf{C}$, $|z|^2 = z\overline{z}$ where $\overline{z}$ is the complex conjugate of z.*

Problem 10. *(20 points) A representation* $\rho : G \to \mathrm{GL}(V)$ *is said to be* ***faithful*** *if* $\ker(\rho) = \{1\}$*. Suppose that* G *is a finite group that has a faithful irreducible representation. Prove that the center of* G *is cyclic.* ***Hint****: Apply Schur's lemma to the elements* $\rho(z)$ *for* z *in the center of* G*.*

Problem 11. ***Bonus*** *(10 points) Suppose G is a finite abelian group, and for every representation $\rho : G \to \mathrm{GL}(V)$, one has that the character $\chi_\rho(g)$ lands in the real numbers $\mathbf{R} \subseteq \mathbf{C}$. Prove that $g^2 = 1$ for every $g \in G$.*

Problem 12. ***Bonus*** *(10 points) Suppose $K \subseteq \mathbf{C}$ is a subfield. Say that K is* ***constructible*** *if there exists a sequence of fields $\mathbf{Q} = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$ such that K_{j+1} is a quadratic extension of K_j for $j = 0, 1, 2, \ldots, n-1$. Prove that $K = \mathbf{Q}(e^{2\pi i/17})$ is constructible.* ***Hint:*** *What is the Galois group of $K/\mathbf{Q}$?*

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