Exam 2 Practice Problems

Instructions The exam will consist of 5 questions. The exam will cover material corresponding to the first 7 homeworks, and discussion section for weeks 1-8. You can expect exam questions to be similar to homework questions, discussion questions, and the questions on this worksheet. *Some exam questions may be exactly the same as a homework question, discussion section question, or a question from this worksheet.*

Problem 1. *TRUE OR FALSE: Suppose F is a field of characteristic 0, K/F a finite Galois extension, and $\beta = \beta_1 \in K$. Let $g(x)$ be the irreducible polynomial for β over F, and let $\beta_1, \dots, \beta_r$ be its roots in K. Then $Gal(K/F)$ necessarily acts transitively on the set $\{\beta_1, \dots, \beta_r\}$.*

Problem 2. *Suppose $f(x) \in \mathbf{Q}[x]$ is a monic irreducible cubic polynomial. Let $\alpha \in \mathbf{C}$ be a root of f, and suppose there exists $\zeta_n = e^{2\pi i/n}$ so that $\alpha \in \mathbf{Q}(\zeta_n)$. Prove that the discriminant of f must be a square in $\mathbf{Q}^\times$.*

Problem 3. *Suppose p is an odd prime number, and let $\zeta_p = e^{2\pi i/p} \in \mathbf{C}$. What is the degree of $[\mathbf{Q}(\zeta_p + \zeta_p^{-1}) : \mathbf{Q}]$?*

Problem 4. *(This problem is too long for an exam question, but I hope it helps your understanding.) The purpose of this question is to develop the basics of* Kummer theory. *Throughout this problem, F is a field of characteristic 0.*

1. *For a field F, and an integer n, let $\mu_n(F) = \{\zeta \in F^\times : \zeta^n = 1\}$ be the group of n^{th} roots of unity in F. Suppose n is an integer, and $\mu_n(F)$ has size n. Let $a \in F^\times$, and let $K = F(a^{1/n})$ be the field obtained by adjoining a root of $x^n - a$ to F. Prove that K/F is Galois.*
2. *Let $\alpha = a^{1/n} \in K$. Define a map of sets $Gal(K/F) \to \mu_n(F)$ as $\sigma \mapsto \sigma(\alpha)/\alpha$. Prove that σ is a well-defined, injective group homomorphism. Deduce that $Gal(K/F)$ is abelian.*
3. *Suppose now $n = p$ is prime, $\mu_p(F)$ has size p, and suppose $a \notin (F^\times)^p$. Prove that $K = F(a^{1/p})$ is a degree p extension of F. **Hint**: $Gal(K/F)$ is a subgroup of $\mu_p(F)$, which is cyclic of order p.*
4. *Conversely, suppose p is prime, $\mu_p(F)$ has size p, and K/F is Galois of degree p. Prove that $K = F(\alpha)$ for some $\alpha \in K$ with $\alpha^p \in F^\times$. **Hint**: Just read the proof of Theorem 16.11.1 in the text.*

Problem 5. *(This problem assumes you have solved the previous one, and is a bit hard.) Let $F = \mathbf{Q}(\zeta_p)$, and suppose $a \in \mathbf{Q}^\times$ but $a \notin (F^\times)^p$. Let $E = F(a^{1/p})$. Prove that $E/\mathbf{Q}$ is Galois. Moreover, prove that $Gal(E/\mathbf{Q})$ is isomorphic to the semidirect product $\mu_p(F) \rtimes (\mathbf{Z}/p\mathbf{Z})^\times$.*

Problem 6. *Give examples (with proof) of primitive elements for each of the following field extensions K/F:*

1. *$F = \mathbf{Q}$, $K = \mathbf{Q}(2^{1/3}, e^{2\pi i/3})$.*
2. *$F = \mathbf{Q}$, $K = \mathbf{Q}(\sqrt{3}, \sqrt{7})$.*

Problem 7. *Let $\mathbf{F}_q$ denote the finite field of size q, if q is a power of a prime number. How many subfields does $\mathbf{F}_{3^{10}}$ contain?*

Problem 8. *Find the irreducible polynomial for $\sqrt{2} + \sqrt{3}$ over $\mathbf{Q}$.*