Exam 2: Math 100C, Spring 2024

You have 60 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT,
(unless a problem specifies otherwise)

Name ______________________________

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ______________________________

There are 5 regular problems, worth 20 points each, and one bonus problem, worth 10 points.

Good luck!

Problem 1. *(20 points) Recall that you proved on a homework question that if* $a, b \in \mathbf{Q}^\times$ *with* $a \notin (\mathbf{Q}^\times)^2, b \notin (\mathbf{Q}^\times)^2$ *and* $ab \notin (\mathbf{Q}^\times)^2$*, then* $\mathbf{Q}(\sqrt{a}, \sqrt{b})$ *is Galois over* $\mathbf{Q}$ *with Galois group isomorphic to the Klein four group. Suppose* $\gamma = w + x\sqrt{a} + y\sqrt{b} + z\sqrt{ab}$*, with* $w, x, y, z \in \mathbf{Q}$*. Write down explicitly (in terms of* a, b, w, x, y, z*) the four elements* $\sigma(\gamma)$ *where* σ *ranges over elements of* $Gal(\mathbf{Q}(\sqrt{a}, \sqrt{b})/\mathbf{Q})$*. You do not need to justify your answer for this question.*

Problem 2. *(20 points) For a positive integer n, let $\zeta_n = e^{2\pi i/n}$. Prove that $2^{1/3}$ is not contained in $\mathbf{Q}(\zeta_n)$ for any n.* ***Hint:*** *We proved in class that $Gal(\mathbf{Q}(\zeta_n)/\mathbf{Q})$ is abelian. You may use this fact.*

Problem 3. *(20 points) Suppose F is a field of characteristic 0, and K is a finite Galois extension of F, with $Gal(K/F) = G$. If $\alpha \in K$, set $N(\alpha) = \prod_{\sigma \in G} \sigma(\alpha)$, the product over the elements $\sigma(\alpha)$ of K, as σ ranges over $G = Gal(K/F)$. Prove that $N(\alpha) \in F$ for all $\alpha \in K$.* ***Hint:*** *Recall that $K^G = F$.*

Problem 4. *(20 points) Let p be a prime number, and suppose F is a finite field of characteristic p. Suppose $f(x) = a_n x^{np} + a_{n-1} x^{(n-1)p} + a_{n-2} x^{(n-2)p} + \cdots + a_1 x^p + a_0 \in F[x]$. Note that the coefficient of x^r in $f(x)$ is 0 unless r is a multiple of p. Prove that there exists a polynomial $g(x) \in F[x]$ so that $f(x) = (g(x))^p$.*

Problem 5. *(20 points) Write TRUE if the statement is true, and write FALSE if the statement is false. For this question, you do not need to justify your response.*

1. *(5 points) Suppose F is a field of characteristic 0, and $f(x) \in F[x]$ is an irreducible monic polynomial of degree 3. Let K denote a splitting field of f. Then $Gal(K/F)$ is cyclic of order 3 if and only if the discriminant of $f(x)$ is in $(F^{\times})^2$.*

2. *(5 points) Suppose $K \supseteq L \supseteq F$ are a tower of fields of characteristic 0. If K/L is Galois and L/F is Galois then necessarily K/F is Galois.*

3. *(5 points) Suppose F_1, F_2 are finite fields of size 27. Then there is exactly one isomorphism of fields $\varphi : F_1 \to F_2$.*

4. *(5 points) Suppose F is a field of characteristic 0, and K is a finite Galois extension of F. Let L be an intermediate field of the extension K/F, so that $K \supseteq L \supseteq F$. Then necessarily K is Galois over L.*

Problem 6. ***Bonus*** *(10 points) Given an example of a subfield K of $\mathbf{C}$ so that $K/\mathbf{Q}$ is a finite Galois extension, with $Gal(K/\mathbf{Q})$ cyclic of order 8.*

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