Exam 1 Practice Problems

Instructions The exam will consist of 5 questions. The exam will cover material corresponding to the first 3 homeworks, and discussion section for weeks 1-4. You can expect exam questions to be similar to homework questions, discussion questions, and the questions on this worksheet. *Some exam questions may be exactly the same as a homework question, discussion section question, or a question from this worksheet.*

Problem 1. *Which of the following statements are true?*

1. *Suppose F is a field, K is an extension field of F, and the dimension of K as an F vector space is finite. Then every element of K is algebraic over F.*
2. *Suppose F is a field, V is a finite-dimensional F vector space, and suppose $T : V \to V$ is a linear map. Then if T is surjective, T is necessarily an isomorphism.*
3. *Suppose F_1, F_2 are two extensions of $\mathbf{Q}$ contained in $\mathbf{C}$, with the property that $F_1 \cap F_2 = \mathbf{Q}$. Assume that $[F_1 : \mathbf{Q}] = 3 = [F_2 : \mathbf{Q}]$. Let K be the subfield of $\mathbf{C}$ generated by F_1 and F_2. Then the degree $[K : \mathbf{Q}]$ is necessarily equal to 9.*
4. *Every quadratic extension K of $\mathbf{Q}$ can be written in the form $K = \mathbf{Q}(\sqrt{d})$ where $d \in \mathbf{Q}$ is nonzero.*

Proof. Parts 1,2,4 are TRUE and 3 is FALSE. □

Problem 2. *Suppose $f(x), g(x) \in \mathbf{Q}[x]$ are monic irreducible polynomials of degrees m and n respectively. Let α be a root of $f(x)$ in $\mathbf{C}$ and β a root of $g(x)$ in $\mathbf{C}$. Set $F = \mathbf{Q}(\alpha)$ and $K = \mathbf{Q}(\beta)$. Suppose that $gcd(m, n) = 1$. Prove that $f(x)$ is irreducible in $K[x]$.*

Proof. The degree of the extension $\mathbf{Q}(\alpha, \beta)$ over $\mathbf{Q}$ must be mn. Thus $\mathbf{Q}(\alpha, \beta)$ has degree m over $K = \mathbf{Q}(\beta)$. Thus the irreducible polynomial $p(x)$ for α over K has degree m. As $f(\alpha) = 0$, $p|f$ in $K[x]$. But both are monic and degree m, so $p = f$ is irreducible in $K[x]$. □

Problem 3. *Let V be a finite-dimensional vector space over the field F, of dimension n. Let $b_1, \dots, b_n$ be a basis of V, $T : V \to V$ a linear map, and $b'_1, \dots, b'_n$ the basis of V^* dual to $b_1, \dots, b_n$. Recall that the trace of T, $\mathrm{tr}(T)$, is defined as the sum of the diagonal entries of the matrix of T with respect to any basis. Prove that $\mathrm{tr}(T) = \sum_{j=1}^n b'_j(T(b_j))$.*

Proof. Let $M = (m_{ij})$ be the matrix of T with respect to the basis $B = (b_1, \dots, b_n)$. Then $T(b_j) = \sum_i m_{ij} b_i$. Consequently $b'_j(T(b_j)) = m_{jj}$. Summing over j gives $\mathrm{tr}(M)$. □

Problem 4. *Suppose F is a field, V an n-dimensional F vector space, and $T : V \to V$ a linear map. Suppose that $\mathrm{tr}(TX) = 0$ for all linear maps $X : V \to V$. Prove that $T = 0$.*

Proof. It is perhaps easy to think about this question in the world of matrices, after picking a basis. Indeed, pick a basis $B = (b_1, \dots, b_n)$ of V. Let $m(T)$, $m(X)$ be the matrices of T and X with respect to this basis B. Then $m(TX) = m(T)m(X)$. Consequently, $\mathrm{tr}(TX) = \mathrm{tr}(m(TX)) = \mathrm{tr}(m(T)m(X)) = \sum_{i,j} m(T)_{ij} m(X)_{ji}$. We can now let X be the linear transformation with matrix $E_{r,s}$: $m(X)_{r,s} = 1$ and other entries equal to 0. Then $\mathrm{tr}(TX) = m(T)_{s,r}$. But this is 0 for all r, s, so $m(T) = 0$ and thus $T = 0$.

Of course, one can also give a proof without using matrices. For example: let $B = (b_1, \dots, b_n)$ be a basis of V. Let $E_{r,s}$ be the linear map $V \to V$ that sends b_s to b_r and is 0 on all other basis vectors. Then

$$\mathrm{tr}(TE_{rs}) = \sum_j b'_j(TE_{r,s}(b_j)) = b'_s(T(b_r)) = 0$$

for all r, s. Since this is 0 for all s, $T(b_r) = 0$. Since this is 0 for all r, $T(b_r) = 0$ for all r. Thus $T = 0$. □

Problem 5. *Suppose F is a field, V an n-dimensional F vector space, and $N : V \to V$ a linear map. The linear map N is said to be* nilpotent *if there exists a positive integer k so that $N^k = 0$. Suppose N is nilpotent.*

1. *Prove that there exists a nonzero vector $b \in V$ so that $N(b) = 0$.*
2. *Prove that $N^n = 0$ on V. **Hint:** Let $V' = Im(N)$ be the image of N. Observe that N preserves V', i.e, if $v' \in V'$ then $N(v') \in V'$. By the rank-nullity theorem, $\dim(V') \leq n - 1$. Now induct on the dimension of V.*

Proof. If N were an isomorphism, N^k would be an isomorphism, and thus would not have a kernel. Thus N is not injective, so there exists $b \in V$, $b \neq 0$, with $N(b) = 0$. This proves the first part. For the second part, by the Rank-Nullity theorem, $\dim(V') \leq n - 1$. By induction, if $v' \in V'$, then $N^{n-1}v' = 0$. Consequently, if $v \in V$, then $v' = N(v)$ satisfies $N^{n-1}v' = 0$, so $N^n v = 0$. □

Problem 6. *Recall that a complex number α is said to be* algebraic *if there exists a monic polynomial $f(x) \in \mathbf{Q}[x]$ for which $f(\alpha) = 0$. Set $\alpha = e^{2\pi i/7} + 2^{1/3} - \sqrt{-3}$.*

1. *Explain how one knows α is algebraic.*
2. *More precisely, prove that there exists a monic polynomial $f(x) \in \mathbf{Q}[x]$ of degree less than* 50 *so that $f(\alpha) = 0$.*

Proof. Each of $e^{2\pi i/7}$, $2^{1/3}$ and $\sqrt{-3}$ are algebraic. As the algebraic numbers form a field, α is algebraic. More precisely, the degree of $e^{2pii/7}$ over $\mathbf{Q}$ is at most 7 (in fact, we proved it is 6), the degree of $2^{1/3}$ over $\mathbf{Q}$ is 3, and the degree of $\sqrt{-3}$ over $\mathbf{Q}$ is 2. Thus $\mathbf{Q}(e^{2\pi i/7}, 2^{1/3}, \sqrt{-3})$ has degree at most $7 \cdot 3 \cdot 2 = 42$ over $\mathbf{Q}$, so α satisfies a polynomial equation with $\mathbf{Q}$ coefficients of degree at most 42. □

Problem 7. *Suppose that p, q are prime numbers, and set $K = \mathbf{Q}(e^{2\pi i/p}, q^{1/p})$, a subfield of $\mathbf{C}$. What is the degree of K over $\mathbf{Q}$?*

Proof. We proved in class that $e^{2\pi i/p}$ has degree $p - 1$ over $\mathbf{Q}$. By Eisenstein's criterion, $q^{1/p}$ has degree p over $\mathbf{Q}$. Since $gcd(p - 1, p) = 1$, K has degree $p(p - 1)$ over $\mathbf{Q}$. □

Problem 8. *Suppose that $\alpha, \beta, \gamma \in \mathbf{C}$. Set $c_1 = \alpha + \beta + \gamma$, $c_2 = \alpha\beta + \beta\gamma + \gamma\alpha$ and $c_3 = \alpha\beta\gamma$. Suppose that the c_j are each algebraic. Must α, β, γ each be algebraic? Be sure to justify your response.*

Proof. The elements α, β, γ are the roots of the polynomial $x^3 - c_1x^2 + c_2x - c_3 \in \overline{\mathbf{Q}}[x]$. Thus, they are indeed algebraic. □

Problem 9. *Let* $F = \mathbf{Q}(\sqrt{2})$*, and let* $a = -1 \in F^\times$*. Set* $\zeta_8 = e^{2\pi i/8}$ *to be a fourth root of* a *in* $\mathbf{C}$*. What is the degree of* $\mathbf{Q}(\sqrt{2}, \zeta_8)$ *over* $\mathbf{Q}(\sqrt{2})$*? (This problem shows that, even though* a *is not a square in* $F^\times$*, taking a fourth root of* a *only gives a quadratic extension of* F*.)*

Proof. We have $\zeta_8 = \sqrt{2}/2 + i\sqrt{2}/2$. So $\mathbf{Q}(\sqrt{2}, \zeta_8) = \mathbf{Q}(\sqrt{2}, i)$ has degree 2 over $\mathbf{Q}(\sqrt{2})$, since -1 is not a square in $\mathbf{Q}(\sqrt{2})$. Alternatively, one can factor $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$. □

Problem 10. *This problem has three parts. (This is too long for an exam question, but hopefully helps your understanding.)*

1. *Suppose* K *is a field,* L *is some (large) extension field of* K*, and* $\delta \in L$ *satisfies* $\delta^2 \in K$*. Let* $a = \delta^2 \in K$*. Explain that if* a *is not a square in* K*, then* $K(\delta) \subseteq L$ *is a quadratic extension of* K*.*
2. *Suppose* $F \subseteq \mathbf{C}$ *is a subfield, and neither* a *nor* $-a$ *is a square in* F*. Let* γ *denote some* 4^{th} *root of* a *in* $\mathbf{C}$ *and set* $L = F(\gamma)$*, a subfield of* $\mathbf{C}$*. Prove that* $[L : F] = 4$*.* ***Hint****. Set* $\delta = \gamma^2$*. Then* $F \subseteq F(\delta) \subseteq F(\gamma)$*. Because* a *is not a square in* F*,* $[F(\delta) : F] = 2$*. Using that* $-a$ *is not a square in* F*, prove that* $\gamma^2 = \delta$ *is not a square in* $F(\delta)$*, so* $[F(\gamma) : F(\delta)] = 2$*. (Compare with the previous problem to see that it is not sufficient to just assume that* a *is not a square in* F*; some extra condition is needed.)*
3. *Use the above analysis to solve problems 3.6 and 3.7(a) in Artin, Chapter 15.*

Proof. Part 1 we did in class. For part 2, we must check that δ is not a square in $F(\delta)$. So, assume $\delta = (u + v\delta)^2$ with $a, b \in F$. Then expanding we see $u^2 + v^2\delta^2 = u^2 + av^2 = 0$ in F, because $(1, \delta)$ is a basis of $F(\delta)$ over F. If $v = 0$ then $u = 0$, a contradiction. If $v \neq 0$, we get $-a = (u/v)^2$, contradicting that $-a$ is a square in F. Thus δ is not a square in F, so $[F(\gamma) : F(\delta)] = 2$. This proves the second part.

Finally, Artin 3.6 is immediate from part 2. For Artin 3.7(a), the answer is that no; $\sqrt{-1}$ is not in $\mathbf{Q}((-2)^{1/4})$. If it were, then the degree $[\mathbf{Q}(i, (-2)^{1/4}) : \mathbf{Q}(i)]$ would have to be 2, by multiplicativity of field degree. But by part 2, one knows this degree is 4 once one checks that ± 2 is not a square in $\mathbf{Q}(i)$. □