Exam 1 Practice Problems

Instructions The exam will consist of 5 questions. The exam will cover material corresponding to the first 3 homeworks, and discussion section for weeks 1-4. You can expect exam questions to be similar to homework questions, discussion questions, and the questions on this worksheet. *Some exam questions may be exactly the same as a homework question, discussion section question, or a question from this worksheet.*

Problem 1. *Which of the following statements are true?*

1. *Suppose F is a field, K is an extension field of F, and the dimension of K as an F vector space is finite. Then every element of K is algebraic over F.*
2. *Suppose F is a field, V is a finite-dimensional F vector space, and suppose $T : V \to V$ is a linear map. Then if T is surjective, T is necessarily an isomorphism.*
3. *Suppose F_1, F_2 are two extensions of $\mathbf{Q}$ contained in $\mathbf{C}$, with the property that $F_1 \cap F_2 = \mathbf{Q}$. Assume that $[F_1 : \mathbf{Q}] = 3 = [F_2 : \mathbf{Q}]$. Let K be the subfield of $\mathbf{C}$ generated by F_1 and F_2. Then the degree $[K : \mathbf{Q}]$ is necessarily equal to 9.*
4. *Every quadratic extension K of $\mathbf{Q}$ can be written in the form $K = \mathbf{Q}(\sqrt{d})$ where $d \in \mathbf{Q}$ is nonzero.*

Problem 2. *Suppose $f(x), g(x) \in \mathbf{Q}[x]$ are monic irreducible polynomials of degrees m and n respectively. Let α be a root of $f(x)$ in $\mathbf{C}$ and β a root of $g(x)$ in $\mathbf{C}$. Set $F = \mathbf{Q}(\alpha)$ and $K = \mathbf{Q}(\beta)$. Suppose that $gcd(m, n) = 1$. Prove that $f(x)$ is irreducible in $K[x]$.*

Problem 3. *Let V be a finite-dimensional vector space over the field F, of dimension n. Let $b_1, \ldots, b_n$ be a basis of V, $T : V \to V$ a linear map, and $b'_1, \ldots, b'_n$ the basis of V^* dual to $b_1, \ldots, b_n$. Recall that the trace of T, $\mathrm{tr}(T)$, is defined as the sum of the diagonal entries of the matrix of T with respect to any basis. Prove that $\mathrm{tr}(T) = \sum_{j=1}^{n} b'_j(T(b_j))$.*

Problem 4. *Suppose F is a field, V an n-dimensional F vector space, and $T : V \to V$ a linear map. Suppose that $\mathrm{tr}(TX) = 0$ for all linear maps $X : V \to V$. Prove that $T = 0$.*

Problem 5. *Suppose F is a field, V an n-dimensional F vector space, and $N : V \to V$ a linear map. The linear map N is said to be* nilpotent *if there exists a positive integer k so that $N^k = 0$. Suppose N is nilpotent.*

1. *Prove that there exists a nonzero vector $b \in V$ so that $N(b) = 0$.*
2. *Prove that $N^n = 0$ on V. **Hint:** Let $V' = Im(N)$ be the image of N. Observe that N preserves V', i.e, if $v' \in V'$ then $N(v') \in V'$. By the rank-nullity theorem, $\dim(V') \leq n - 1$. Now induct on the dimension of V.*

Problem 6. *Recall that a complex number α is said to be* algebraic *if there exists a monic polynomial $f(x) \in \mathbf{Q}[x]$ for which $f(\alpha) = 0$. Set $\alpha = e^{2\pi i/7} + 2^{1/3} - \sqrt{-3}$.*

1. *Explain how one knows α is algebraic.*
2. *More precisely, prove that there exists a monic polynomial $f(x) \in \mathbf{Q}[x]$ of degree less than 50 so that $f(\alpha) = 0$.*

Problem 7. *Suppose that p, q are prime numbers, and set $K = \mathbf{Q}(e^{2\pi i/p}, q^{1/p})$, a subfield of $\mathbf{C}$.*

What is the degree of K over $\mathbf{Q}$?

Problem 8. *Suppose that $\alpha, \beta, \gamma \in \mathbf{C}$. Set $c_1 = \alpha + \beta + \gamma$, $c_2 = \alpha\beta + \beta\gamma + \gamma\alpha$ and $c_3 = \alpha\beta\gamma$. Suppose that the c_j are each algebraic. Must α, β, γ each be algebraic? Be sure to justify your response.*

Problem 9. *Let $F = \mathbf{Q}(\sqrt{2})$, and let $a = -1 \in F^\times$. Set $\zeta_8 = e^{2\pi i/8}$ to be a fourth root of a in $\mathbf{C}$. What is the degree of $\mathbf{Q}(\sqrt{2}, \zeta_8)$ over $\mathbf{Q}(\sqrt{2})$? (This problem shows that, even though a is not a square in $F^\times$, taking a fourth root of a only gives a quadratic extension of F.)*

Problem 10. *This problem has three parts. (This is too long for an exam question, but hopefully helps your understanding.)*

1. *Suppose K is a field, L is some (large) extension field of K, and $\delta \in L$ satisfies $\delta^2 \in K$. Let $a = \delta^2 \in K$. Explain that if a is not a square in K, then $K(\delta) \subseteq L$ is a quadratic extension of K.*
2. *Suppose $F \subseteq \mathbf{C}$ is a subfield, and neither a nor $-a$ is a square in F. Let γ denote some 4^{th} root of a in $\mathbf{C}$ and set $L = F(\gamma)$, a subfield of $\mathbf{C}$. Prove that $[L : F] = 4$. **Hint**. Set $\delta = \gamma^2$. Then $F \subseteq F(\delta) \subseteq F(\gamma)$. Because a is not a square in F, $[F(\delta) : F] = 2$. Using that $-a$ is not a square in F, prove that $\gamma^2 = \delta$ is not a square in $F(\delta)$, so $[F(\gamma) : F(\delta)] = 2$. (Compare with the previous problem to see that it is not sufficient to just assume that a is not a square in F; some extra condition is needed.)*
3. *Use the above analysis to solve problems 3.6 and 3.7(a) in Artin, Chapter 15.*