

Friday May 26

Finitely gen'd abln gps

Thm: Suppse V is a f.g. abln gp. Then

$$\boxed{V \cong \underbrace{C_{d_1}} \oplus \underbrace{C_{d_2}} \oplus \dots \oplus \underbrace{C_{d_k}}}_{\text{where } d_1 | d_2 | d_3 | \dots | d_k} \oplus \mathbb{Z}^r$$

Recall: If $(m, n) = 1$, $C_{mn} \cong C_m \oplus C_n$ (CRT)

Corollary (Alt form of structure thm) Every fte abln gp is a direct sum of cyclic gps of prime power order

Pf: If $m = p_1^{e_1} \dots p_r^{e_r}$, then

$$\underline{C_m \cong C_{p_1^{e_1}} \oplus \dots \oplus C_{p_r^{e_r}}} \quad \textcircled{a}$$

Uniqueness Thm Suppse a fte abln gp V is a direct sum of cyclic gps of prime power orders $d_j = p_j^{r_j}$. Then the d_j are uniquely determined by V . I.e., if

$$V \cong C_{d_1} \oplus \dots \oplus C_{d_l} \cong C_{e_1} \oplus \dots \oplus C_{e_k}$$

if d_i, e_j prime powers, then $l=k$ and $d_j = e_j \forall j$ up to rearrangement!

Example $C_{16}, C_4 \oplus C_4, C_2 \oplus C_2 \oplus C_4$

$$\begin{array}{ccccccc} & & 2 & 2 & 2 & 2 & 2 \\ & & 2 & 2 & 2 & 2 & 2 \\ & & 1 & 1 & 1 & 1 & 1 \end{array} \rightsquigarrow 8 \text{ choices}$$

4 choices

($2x \equiv 0 \pmod{16}$
 $\Rightarrow x \equiv 0 \pmod{8}$)

Pf: • Suppose p is a prime that occurs in the decomposition of V into cyclic gps of prime power order, i.e., $d_i = p^{l_i}$ for some i .

• c_i the # of cyclic gps of order p^{l_i} in the decomposition

• $V[p^i] = \{v \in V : p^i v = 0\} \leq V$, a subgp.

$$\# \underline{V[p^i]} = p^{l_i c_i}, \text{ some } l_i \in \mathbb{Z}_{\geq 0}$$

• k the largest index s.t. $c_k > 0$.

THEN

$$l_1 = c_1 + c_2 + c_3 + \dots + c_k$$

$$l_2 = c_1 + 2c_2 + 2c_3 + \dots + 2c_k$$

($n^2 \cdot 0 \quad 127$)

$$\begin{cases}
 l_2 = c_1 + 2c_2 + 2c_3 + \dots + 2c_k \\
 \text{b/c } \#C_{p^j}[p^2] = \begin{cases} p^2 & \text{if } j \geq 2 \\ p_j & \text{if } j < 2 \end{cases} \\
 l_3 = c_1 + 2c_2 + 3c_3 + \dots + 3c_k \\
 \vdots \\
 l_k = c_1 + 2c_2 + 3c_3 + \dots + kc_k \\
 \text{b/c } \left(\#C_{p^j}[p^l] = \begin{cases} p^l & \text{if } j \geq l \\ p_j & \text{if } j < l \end{cases} \right)
 \end{cases}$$

The l 's are clearly uniquely determined from V .

WTS: the c 's are .

Have
$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_k \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

Matrix has $\det = 1 \Rightarrow c$'s are determined.

Example $k=4$

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \det = 1 \text{ by induction}$$

⑩

Cor: Suppose $V \cong C_{d_1} \oplus \dots \oplus C_{d_k}$ w/ $d_i > 1$ and $d_1 | d_2 | \dots | d_k$. Then the d_j are uniquely determined

Pf:

$$\begin{aligned} d_1 &= p_1^{e_{11}} p_2^{e_{12}} \dots p_r^{e_{1r}} \\ d_2 &= p_1^{e_{21}} p_2^{e_{22}} \dots p_r^{e_{2r}} \\ &\vdots \\ d_k &= p_1^{e_{k1}} p_2^{e_{k2}} \dots p_r^{e_{kr}} \end{aligned} \Rightarrow V \cong \bigoplus_{\substack{1 \leq i \leq k \\ 1 \leq j \leq r}} C_{p_i^{e_{ij}}} \quad (*)$$

$$\text{w/ } 0 \leq e_{1j} \leq e_{2j} \leq \dots \leq e_{kj} \quad \forall j.$$

So, if you list all the prime powers that appear in $(*)$, d_k is the product of the "largest" ones, i.e. the powers of the $p_1, \dots, p_r$ w/ the largest exponents.

Similarly, d_{k-1} is the product of the next largest ones, and so on.

Thus, the d_k are uniquely determined. ⑩

• $k = \text{field}$

• Let $E_n^{k[T]} = \langle D_n, S_n, E_{ij}(a) \rangle \subseteq GL_n(k[T])$
 $i \neq j$
 $a \in k[T]$

where $D_n = \begin{pmatrix} k^x & & \\ & k^x & \\ & & \ddots \\ & & & k^x \end{pmatrix}$

Thm: Suppose $A \in M_{m,n}(k[T])$. Then $\exists P \in E_n^{k[T]}$
 $Q \in E_m^{k[T]}$ st. $Q^{-1}AP$ is diagonal with entries
 $d_1, d_2, \dots$ where $d_i \in k[T]$ monic & $d_1 | d_2 | d_3 | \dots$.

Pf: $k[T]$ is a Euclidean domain, so proof carries over.

Cor: The inclusion $E_n^{k[T]} \subseteq GL_n(k[T])$ is an equality.

Def: R any ring. A cyclic R -mod C is a module generated by a single elt v . Thus $\cong R/I$ for some ideal I . (Pf: $R \twoheadrightarrow C \Rightarrow R/I \cong C$.)
 $r \mapsto rv$

So, if $R = k[T]$ (or a PID), then $C \cong R/(d)$, $d \in R$.

Thm Let k be a field, and $R = k[T]$. Suppose V is a f.g. R -mod.

a) Then $V \cong C_{d_1} \oplus C_{d_2} \oplus \dots \oplus C_{d_k} \oplus L$ where

$C_d \cong R/(d)$, L is free of finite rank, and $d_1 | d_2 | \dots | d_k$.

b) Same conclusion as (a), but replace the d_i w/ $p_i^{e_i}$, p_i monic irreducible

Pf: Same as before.

More generally, Thm holds for any PID, and the d_i 's are unique in either decomposition

Applications to linear operators

Observe: If V is a fin. dim'd space (k , $T: V \rightarrow V$ a linear operator), then V is a $k[x]$ -module via $x \cdot v = T(v)$.

Conversely, if V is a $k[x]$ -module, then V is a k space w/ a linear operator: namely

the space w/ a linear operator, namely,

$$T: V \rightarrow V \text{ as}$$

$$T(v) = \lambda \cdot v$$

V is f.g., ($\Rightarrow$) V is f.g.d and has no "free part" in its decomposition.

Observe: If $V = C_d$ $d = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in k[X]$,
then V has basis $1, X, X^2, \dots, X^{n-1}$.

T has matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & \vdots \\ \vdots & 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & 1 & -a_{n-1} \end{pmatrix}$$

Cor: If $T: V \rightarrow V$ is a lin operator, V finite dim'd k space, then there is a basis of V s.t.

T has block form in this basis, w/ blocks of the type above.

Pf: $V \cong C_{d_1} \oplus \dots \oplus C_{d_k}$ as above.

The form of this matrix is called "rational canonical form".