

Lecture for Wed May 24

Noetherian rings

R , ring, is noetherian if every ideal $I \subseteq R$ is generated by finitely many elts.

E.g. $\mathbb{Z}$, k , $k[x]$ are Noetherian

[Thm: Suppose R is noetherian, V f.g. R -mod.
Then every submod of V is f.g.]

Non-example. $R = k[x_1, x_2, x_3, \dots]$ (infinitely # of variables)

• $V = R$ (free of rank 1)

• $I = (x_1, x_2, x_3, \dots)$: I is not f.g.

Ex of Thm: $R = \mathbb{Z}$, $V = \mathbb{Z}^m$, $U \subseteq V$ subg.

Then U is f.g.

We used this to prove that $U \cong \mathbb{Z}^k$, $k \leq m$

Propⁿ: R noetherian, $J \subseteq R$ ideal, R/J is noetherian.

Thm (HBT) R noetherian $\Rightarrow R[x]$ is noetherian.

Lemma: R ring, $I \subseteq R[x]$ an ideal

$$A = \{a \in R: \underbrace{ax^n + \dots}_{\text{L.O.T.}} \in I\} \ni 0$$

Thm: $A \subseteq R$ is an ideal.

Pf: $a, b \in A$.

$$\Rightarrow I \ni f = ax^n + \dots$$

WLOG, $n \geq m$.

$$\ni g = bx^m + \dots$$

$$f + x^{n-m}g = (a+b)x^n + \dots \Rightarrow a+b \in A$$

$$\text{Also, if } r \in I \Rightarrow ra \in A. \quad \square$$

Pf of HBT: $I \subseteq R[x]$, NTS: I is f.g.

A = ideal of leading coeffs of elts of I

$$R \text{ noeth} \Rightarrow A = (r_1, \dots, r_k).$$

Can Assume, WLOG,

$$f_1 = r_1 x^n + \dots \in I$$

$$f \in I$$

$$\begin{aligned}
 f_1 &= \alpha_1 X^n + \dots \\
 f_2 &= \alpha_2 X^n + \dots \in I \\
 &\vdots \\
 f_k &= \alpha_k X^n + \dots \in I
 \end{aligned}$$

$$P = R \cdot 1 + R \cdot X + R \cdot X^2 + \dots + R \cdot X^{n-1} \subseteq R[X]$$

$$\cong R^n \text{ as } R\text{-mod}$$

$$\underline{W} = I \cap P \subseteq P \text{ as an } R \text{ submod}$$

$$R \text{ noeth} \Rightarrow W \text{ is f.g. as an } R \text{ mod.}$$

$$W = (\underline{h_1, \dots, h_l})$$

$$\underline{\text{Claim:}} \quad I = (\underline{h_1, \dots, h_l}, f_1, \dots, f_k) \text{ as ideals.}$$

$$\underline{\text{Pf of claim:}} \quad \text{Suppose } g \in I$$

$$\text{If } \deg(g) < n, \text{ done}$$

$$\text{If } \deg(g) \geq n, \text{ say } g = \alpha X^d + \dots, \quad d \geq n$$

$$\alpha = r_1 \alpha_1 + \dots + r_k \alpha_k$$

$$\Rightarrow \underbrace{g - (r_1 f_1 + \dots + r_k f_k)}_{\in I} X^{d-n} \text{ has } \deg < d.$$

$$\text{Induct on } \deg(g).$$

Structure of Abelian Grps

Direct Σ of R -mods

$V: R \text{ mod}, W_1, \dots, W_k \text{ submods.}$

$$\cdot W_1 + \dots + W_k = \{v \in V: v = w_1 + \dots + w_k: w_j \in W_j\}$$

• The W_j 's are independent if

$$w_1 + \dots + w_k = 0 \Rightarrow w_j = 0 \quad \forall j. \\ \text{and } w_j \in W_j$$

• One says V is the (internal) direct sum of the W_j 's if

$$\cdot V = W_1 + \dots + W_k$$

• the W_j are independent.

$$V = W_1 \oplus \dots \oplus W_k.$$

$$\underline{k=2} \quad V = W_1 \oplus W_2 \iff V = W_1 + W_2 \text{ and } W_1 \cap W_2 = \{0\}.$$

Def³ (External direct sums) Suppose $W_1, \dots, W_k$ R mods.

$$\text{Then } W_1 \oplus \dots \oplus W_k = \{ (w_1, w_2, \dots, w_k) : w_j \in W_j \}$$

Then $W_1 \oplus \dots \oplus W_k = \{ (w_1, w_2, \dots, w_k) : w_j \in W_j \}$

w/ addition componentwise

$$\lambda(w_1, \dots, w_k) = (\lambda w_1, \dots, \lambda w_k).$$

Internal & External direct sums are equivalent notions.

Propn $V_1, \dots, V_k$ R -mods, $U_j \leq V_j$ submod.

$$\begin{aligned} V/U &:= V_1 \oplus \dots \oplus V_k / U \cong V_1/U_1 \oplus \dots \oplus V_k/U_k \\ U &:= U_1 \oplus \dots \oplus U_k \end{aligned}$$

Pf: $V = V_1 \oplus \dots \oplus V_k \longrightarrow V_1/U_1 \oplus \dots \oplus V_k/U_k$

suggestive? $\checkmark$.

kernel: $v = (v_1, \dots, v_k)$ s.t. $v_j \in U_j \forall j$
 $= U.$

Thm. Structure Thm for Ab^{ln} gp A f.g. ab^{ln} gp V

is isom. to $C_{d_1} \oplus \dots \oplus C_{d_k} \oplus L$ where

- C_{d_j} : cyclic of order d_j
- L free of fte rk
 $\perp d_1, d_2, \dots$

$$d_1 | d_2, d_2 | d_3, \dots$$

Pf: $\mathbb{Z}$ noetherian, V f.g. $\Rightarrow V$ finitely presented.

$$\Rightarrow V \cong \mathbb{Z}^m / \underbrace{A \cdot \mathbb{Z}^n}_{\substack{\text{in} \\ \mathbb{Z}^m}} \quad \text{for some } m, n \in \mathbb{Z}_{\geq 0} \\ A \in M_{n,n}(\mathbb{Z}).$$

Up to changing bases, can diagonalize A .

This means $\exists$ basis $u_1, \dots, u_m$ of $\mathbb{Z}^m$ s.t.

$$A \mathbb{Z}^n = \mathbb{Z} a_1 u_1 \oplus \mathbb{Z} a_2 u_2 \oplus \dots \oplus \mathbb{Z} a_l u_l, \quad l \leq m$$

$$\text{w/ } a_1 | a_2, a_2 | a_3, \dots$$

$$\Rightarrow \mathbb{Z}^m / A \mathbb{Z}^n \cong_{\text{prop}} \mathbb{Z} / a_1 \mathbb{Z} \oplus \mathbb{Z} / a_2 \mathbb{Z} \oplus \dots \oplus \mathbb{Z} / a_l \mathbb{Z} \oplus \mathbb{Z}^{m-l}$$

□