

Final Exam: Math 100C, Spring 2023

You have 3 hours

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name _____

“I have adhered to UCSD policies on academic integrity while completing this examination.”

Signature _____

There are 10 problems; each problem is worth 20 points.

Good luck!

Problem 1. (20 points) Answer TRUE if the statement is true, and FALSE if the statement is false. For this question, you do not need to show your work or justify your response.

1. (5 points) The ring $\mathbf{Z}[x, y, z]/(y^2 - xz)$ is Noetherian.
2. (5 points) Suppose n is a positive integer and $M \subseteq \mathbf{Z}^n$ is a subgroup. Then there exists n elements $m_1, \dots, m_n$ of M so that M is generated by $m_1, \dots, m_n$, i.e., $M = \mathbf{Z}m_1 + \dots + \mathbf{Z}m_n$.
3. (5 points) Let C_n denote the cyclic group of order n . Then $C_2 \times C_4 \times C_8$ is isomorphic to $C_4 \times C_{16}$.
4. (5 points) Suppose F is a field and V is a 10-dimensional F vector space. Let $N : V \rightarrow V$ be a linear map that satisfies $N^{100} = 0$. Then necessarily $N^{10} = 0$.

Problem 2. (20 points) For a prime power q , let $\mathbf{F}_q$ denote the field of order q .

1. (10 points) How many subfields does $\mathbf{F}_{2^6}$ have?

2. (10 points) How many **primitive elements** does $\mathbf{F}_{2^6}$ have?

Problem 3. (20 points) Suppose $f(x) \in \mathbf{Q}[x]$ is a monic irreducible polynomial. Assume in $\mathbf{C}[x]$ the polynomial f factors as $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$, and let $E \subseteq \mathbf{C}$ be the splitting field of $f(x)$ in $\mathbf{C}$.

1. (10 points) Recall that $D(f) = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$ is called the discriminant of the polynomial f . Explain how one knows that $D(f) \in \mathbf{Q}$.

2. (10 points) Let $G = \text{Gal}(E/\mathbf{Q})$, and suppose $D(f)$ is a square in $\mathbf{Q}$, i.e., there exists $\delta \in \mathbf{Q}$ so that $D(f) = \delta^2$. Explain how one knows that in this case, the Galois group G is isomorphic to a subgroup of the alternating group $A_n \subseteq S_n$.

Problem 4. (20 points) Let $E = \mathbf{Q}(i, 2^{1/4}) \subseteq \mathbf{C}$. Observe that, because $i \notin \mathbf{Q}(2^{1/4})$, $[E : \mathbf{Q}] = 8$ and then $[E : \mathbf{Q}(i)] = 4$. (You may use these facts without reproving them.)

1. (5 points) Prove that $\text{Gal}(E/\mathbf{Q}(i))$ is cyclic of order four.

2. (15 points) Prove that $\text{Gal}(E/\mathbf{Q})$ is dihedral of order 8.

For this part of the problem, you may assume the prior part, even if you did not answer the prior part.

Problem 5. (20 points) Let F be a field, V a four-dimensional F vector space, and $T : V \rightarrow V$ a linear operator. Is it possible that there exists a cubic irreducible polynomial $f(x) \in F[x]$ so that $f(T) = 0$ on V ? Be sure to justify your response.

Problem 6. (20 points) Let G denote the abelian group with generators x, y , subject to the relations $x^{100} = 1$, $y^{50} = 1$ and $x^{25} = y^{25}$. Write G as a product of cyclic groups. Be sure to prove your answer.

Problem 7. (20 points) Suppose E_1 and E_2 are Galois extensions of $\mathbf{Q}$ in $\mathbf{C}$ with $\text{Gal}(E_1/\mathbf{Q}) \simeq \text{Gal}(E_2/\mathbf{Q}) \simeq S_3$. Let $K = E_1 \cap E_2$ and assume that K is quadratic over $\mathbf{Q}$. Let E denote the compositum of E_1 and E_2 in $\mathbf{C}$. How many subfields of E are there that contain K ? Be sure to prove your answer.

Problem 8. (20 points) Suppose $K/\mathbf{Q}$ is a Galois extension of degree three. Let $F = \mathbf{Q}(\mu_3)$. We consider both K and F as subfields of $\mathbf{C}$. Let L denote the compositum of K and F in $\mathbf{C}$.

For each part of this problem, you may assume the prior parts, even if you did not answer the prior parts.

1. (5 points) Prove that $L/\mathbf{Q}$ is Galois, and $\text{Gal}(L/\mathbf{Q}) \simeq C_3 \times C_2$. Here C_n denotes the cyclic group of order n .
2. (5 points) Prove that there exists $v \in F^\times$ so that $L = F(v^{1/3})$.
3. (10 points) With v as in the previous part, let $\bar{v}$ denote the complex conjugate of v . Prove that $\bar{v} \equiv v^2 \pmod{(F^\times)^3}$, i.e., that there exists $w \in F^\times$ so that $\bar{v} = v^2 w^3$.

Problem 9. (20 points) Let $n \geq 3$ be an integer and let D_n denote the dihedral group of order $2n$. Prove that every irreducible representation of D_n has dimension 1 or 2.

Problem 10. (20 points) Let C_4 denote the cyclic group of order 4 and let K_4 denote the Klein 4-group.

1. (10 points) Make a character table for C_4 .

2. (10 points) Make a character table for K_4 .

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