

Exam 2: Math 100C, Spring 2023

You have 50 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name _____

“I have adhered to UCSD policies on academic integrity while completing this examination.”

Signature _____

There are 5 problems; each problem is worth 20 points.

Good luck!

Problem 1. (20 points) Suppose p is a prime number, $n > 2$ is an integer, and N is a positive integer. Prove that $\mathbf{Q}(p^{1/n})$ is not contained in $\mathbf{Q}(\mu_N)$.

Proof. (This problem was on the practice worksheet for exam 2.) One notes that $\mathbf{Q}(\mu_N)$ is Galois over $\mathbf{Q}$ with abelian Galois group. Thus, because every subgroup of an abelian group is normal, every subfield of $\mathbf{Q}(\mu_N)$ is Galois over $\mathbf{Q}$. But because $n > 2$, $\mathbf{Q}(p^{1/n})$ is not Galois over $\mathbf{Q}$: it does not contain the other roots of $x^n - p$, for example it does not contain $e^{2\pi i/n} p^{1/n}$. (Recall that if an irreducible polynomial, such as $x^n - p$ which is irreducible by Eisenstein, contains one root in a Galois extension then it splits in that extension.) $\square$

Problem 2. (20 points) For this problem, you do not need to justify your answers.

1. (6 points) Give an example of extension fields L/K and K/F , so that L/K and K/F are finite Galois but L/F is not Galois.

2. (7 points) Give an example of a Galois extension E of $\mathbf{Q}$ so that $\text{Gal}(E/\mathbf{Q}) \simeq S_3 \times C_2$, where S_3 denotes the symmetric group and C_2 denotes the cyclic group of order two.

3. (7 points) Give an example of a Galois extension K of $\mathbf{Q}$ so that $\text{Gal}(K/\mathbf{Q}) \simeq C_2 \times C_2 \times C_2$. Here again C_2 denotes the cyclic group of order 2.

Proof. Here are some examples:

1. $F = \mathbf{Q}$, $K = \mathbf{Q}(3^{1/2})$, $L = \mathbf{Q}(3^{1/4})$. (This question was on the practice worksheet.)
2. For example, we can take $K = \mathbf{Q}(\mu_3, 2^{1/3}, 3^{1/2})$. (This was essentially a homework question.)
3. One can take $K = \mathbf{Q}(2^{1/2}, 3^{1/2}, 5^{1/2})$. (We discussed these sorts of examples in class.)

□

Problem 3. (20 points) Suppose F is a field of characteristic 0, and K is a finite Galois extension of F of degree n . Suppose $\tau \in \text{Gal}(K/F)$, $\gamma \in K^\times$, and suppose $\tau(\gamma) = \lambda\gamma$ for some $\lambda \in F^\times$. Prove that $\lambda^n = 1$. **Hint:** One way to proceed is to consider the norm $N_{K/F}$ of each side of the equation $\tau(\gamma) = \lambda\gamma$. Another way to proceed is to iteratively apply τ to both sides of this equation.

Proof. For the proof via norms, note that you proved in your homework that $N_{K/F}(x) = \prod_{\sigma \in \text{Gal}(K/F)} \sigma(x)$. Thus, $N_{K/F}(\tau(\gamma)) = N_{K/F}(\gamma)$. On the other hand, $N_{K/F}(\lambda\gamma) = \lambda^n N_{K/F}(\gamma)$. Because $\gamma \in K^\times$, $N_{K/F}(\gamma) \in F^\times$, so we may cancel to obtain $\lambda^n = 1$.

For the proof via iteratively applying τ , note that we have $\tau^j(\gamma) = \lambda\tau^{j-1}(\gamma) = \lambda^2\tau^{j-2}(\gamma) = \cdots = \lambda^j\gamma$. Because $\text{Gal}(K/F)$ has order n , $\tau^n = 1$ by Lagrange's theorem, so $\gamma = \lambda^n\gamma$ and thus $\lambda^n = 1$. $\square$

Problem 4. (20 points) For this question, answer TRUE if the statement is true, and FALSE if the statement is false. For this question, you do not need to justify your answers.

1. (5 points) Suppose E is a finite Galois extension of $\mathbf{Q}$, with $\text{Gal}(E/\mathbf{Q})$ isomorphic to the symmetric group S_3 . Then there are always exactly three distinct subfields E_1, E_2, E_3 of E with $[E_1 : \mathbf{Q}] = [E_2 : \mathbf{Q}] = [E_3 : \mathbf{Q}] = 3$.

2. (5 points) Suppose $K_1, K_2, \dots, K_n$ are finite Galois extensions of $\mathbf{Q}$, all contained in $\mathbf{C}$, with each $\text{Gal}(K_j/\mathbf{Q})$ abelian. Let L be the compositum of the K_j in $\mathbf{C}$. Then L is necessarily Galois with abelian Galois group.

3. (5 points) Every finite extension E of $\mathbf{Q}$ has finitely many subfields.

4. (5 points) The field $\mathbf{Q}(\mu_{11})$ has four subfields.

Proof. All statements are TRUE.

□

Problem 5. (20 points) For the two parts to this question, be sure to prove your answer.

1. (10 points) Let K denote the splitting field of $x^5 - 3$ over $\mathbf{Q}$. What is the degree of K over $\mathbf{Q}$?

Proof. We have that K is the compositum of $\mathbf{Q}(3^{1/5})$ (which has degree 5 over $\mathbf{Q}$ by Eisenstein) and $\mathbf{Q}(\mu_5)$ which has degree 4 over $\mathbf{Q}$. Because $\gcd(5, 4) = 1$, K has degree 20. $\square$

2. (10 points) Let E denote the splitting field of $x^4 - 2$ over $\mathbf{Q}$. What is the degree of E over $\mathbf{Q}$?

Proof. $E = \mathbf{Q}(2^{1/4}, i)$. We have $\mathbf{Q}(2^{1/4})$ has degree 4 over $\mathbf{Q}$. Because $\mathbf{Q}(2^{1/4}) \subseteq \mathbf{R}$, i is not in this field, so adjoining i multiplies the degree by 2. Thus E has degree 8 over $\mathbf{Q}$. $\square$

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