

Exam 2: Math 100C, Spring 2023

You have 50 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name _____

“I have adhered to UCSD policies on academic integrity while completing this examination.”

Signature _____

There are 5 problems; each problem is worth 20 points.

Good luck!

Problem 1. (20 points) Suppose p is a prime number, $n > 2$ is an integer, and N is a positive integer. Prove that $\mathbf{Q}(p^{1/n})$ is not contained in $\mathbf{Q}(\mu_N)$.

Problem 2. (20 points) For this problem, you do not need to justify your answers.

1. (6 points) Give an example of extension fields L/K and K/F , so that L/K and K/F are finite Galois but L/F is not Galois.

2. (7 points) Give an example of a Galois extension E of $\mathbf{Q}$ so that $\text{Gal}(E/\mathbf{Q}) \simeq S_3 \times C_2$, where S_3 denotes the symmetric group and C_2 denotes the cyclic group of order two.

3. (7 points) Give an example of a Galois extension K of $\mathbf{Q}$ so that $\text{Gal}(K/\mathbf{Q}) \simeq C_2 \times C_2 \times C_2$. Here again C_2 denotes the cyclic group of order 2.

Problem 3. (20 points) Suppose F is a field of characteristic 0, and K is a finite Galois extension of F of degree n . Suppose $\tau \in \text{Gal}(K/F)$, $\gamma \in K^\times$, and suppose $\tau(\gamma) = \lambda\gamma$ for some $\lambda \in F^\times$. Prove that $\lambda^n = 1$. **Hint:** One way to proceed is to consider the norm $N_{K/F}$ of each side of the equation $\tau(\gamma) = \lambda\gamma$. Another way to proceed is to iteratively apply τ to both sides of this equation.

Problem 4. (20 points) For this question, answer TRUE if the statement is true, and FALSE if the statement is false. For this question, you do not need to justify your answers.

1. (5 points) Suppose E is a finite Galois extension of $\mathbf{Q}$, with $\text{Gal}(E/\mathbf{Q})$ isomorphic to the symmetric group S_3 . Then there are always exactly three distinct subfields E_1, E_2, E_3 of E with $[E_1 : \mathbf{Q}] = [E_2 : \mathbf{Q}] = [E_3 : \mathbf{Q}] = 3$.

2. (5 points) Suppose $K_1, K_2, \dots, K_n$ are finite Galois extensions of $\mathbf{Q}$, all contained in $\mathbf{C}$, with each $\text{Gal}(K_j/\mathbf{Q})$ abelian. Let L be the compositum of the K_j in $\mathbf{C}$. Then L is necessarily Galois with abelian Galois group.

3. (5 points) Every finite extension E of $\mathbf{Q}$ has finitely many subfields.

4. (5 points) The field $\mathbf{Q}(\mu_{11})$ has four subfields.

Problem 5. (20 points) For the two parts to this question, be sure to prove your answer.

1. (10 points) Let K denote the splitting field of $x^5 - 3$ over $\mathbf{Q}$. What is the degree of K over $\mathbf{Q}$?

2. (10 points) Let E denote the splitting field of $x^4 - 2$ over $\mathbf{Q}$. What is the degree of E over $\mathbf{Q}$?

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