

Exam 1: Math 100C, Spring 2023

You have 50 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name _____

“I have adhered to UCSD policies on academic integrity while completing this examination.”

Signature _____

There are 5 problems; each problem is worth 20 points.

Good luck!

Problem 1. (20 points) Let p be a prime number. Give a simple formula in terms of p for the number of elements in the group $\mathrm{GL}_3(\mathbf{F}_p)$. Be sure to prove your answer.

Problem 2. (20 points) Suppose n is a positive integer and p is a prime number which is relatively prime to n . Assume $f(x) \in \mathbf{Z}[x]$ is a monic polynomial of positive degree that divides the polynomial $x^n - 1$ in $\mathbf{Z}[x]$, i.e., there exists a polynomial $g(x) \in \mathbf{Z}[x]$ so that $x^n - 1 = f(x)g(x)$. Prove that the mod p reduction of $f(x)$ has no multiple roots in any field of characteristic p .

Problem 3. (20 points) For this question, answer *TRUE* if the statement is true, and *FALSE* if the statement is false. For this question only, you do not need to show your work or provide an explanation.

1. (5 points) There exists a monic irreducible polynomial $f(x) \in \mathbf{Q}[x]$ of degree 11 which becomes reducible over $\mathbf{Q}[\sqrt{3}]$.

2. (5 points) For every positive integer n , there exists a monic irreducible polynomial in $\mathbf{F}_{11}[x]$ of degree n .

3. (5 points) Set $K = \mathbf{F}_{81}$ and $F = \mathbf{F}_3$. If an element $\gamma \in K$ is a primitive element for the extension K/F , then γ necessarily generates the multiplicative group $K^\times$.

4. (5 points) Suppose $n_1, n_2 \in \mathbf{Z}$ are distinct square-free integers, neither of which is equal to 1. Then $\mathbf{Q}(\sqrt{n_1})$ cannot be isomorphic to $\mathbf{Q}(\sqrt{n_2})$ as fields.

Problem 4. (20 points) Let $p = 3$ and $r = 12$. How many subfields does $K = \mathbf{F}_{p^r} = \mathbf{F}_{3^{12}}$ contain? Be sure to justify your answer.

Problem 5. (20 points) Give an example of a field extension $K/\mathbf{Q}$ so that the degree $[K : \mathbf{Q}] = 5$ but $\text{Aut}(K/\mathbf{Q}) = \{1\}$. Be sure to justify your answer.

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