

Mathematics 100C Homework 5

Due: Wednesday May 10 2023

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Let F be a field. Let V be a finite-dimensional F vector space, and $T : V \rightarrow V$ an F -linear map. Suppose $\alpha_1, \dots, \alpha_n \in F$ are distinct, and $p(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ satisfies $p(T) = 0$ on V . Prove that $V = \bigoplus_j V_{\alpha_j}$, where $V_{\alpha} = \{v \in V : Tv = \alpha v\}$ is the α -eigenspace for T . **Hint:** Set $R = k[x]/(p(x))$. Using the CRT, prove that there exists elements $1_j(x) \in R$ for $1 \leq j \leq n$ so that $1 = \sum_j 1_j(x)$ and $x1_j(x) = \alpha_j 1_j(x)$ in R . Set $W_j = 1_j(T)V$. Prove that $W_j \subseteq V_{\alpha_j}$ and that $V = \sum_j W_j$. Finally, check that the W_j or that the V_{α_j} are linearly independent, to finish the proof.
2. Let F be a field of characteristic 0, and $M \supseteq L \supseteq F$ a tower of fields, for which M/F is Galois. Prove that L/F is Galois if and only if $\sigma(L) \subseteq L$ for all $\sigma \in \text{Gal}(M/F)$.
3. Suppose F_0 is a field of characteristic 0, and F/F_0 is a finite Galois extension with Galois group G . Suppose F contains μ_n , and L/F is a Kummer extension (ie, finite Galois extension with Galois group abelian of exponent n), and set as usual $\Delta(L/F) = ((L^\times)^n \cap F^\times)/(F^\times)^n$. Observe that the Galois group G acts on $F^\times/(F^\times)^n$. Suppose now that $g(\Delta(L/F)) \subseteq \Delta(L/F)$ for every $g \in G$.
 - (a) Prove that L/F_0 is Galois. **Hint:** You may find it helpful to use the previous exercise.
 - (b) Suppose T is a group, $H \leq T$ is a normal subgroup with H is abelian. Check that the conjugation action of T on H induces a well-defined action of $\bar{T} = T/H$ on H .
 - (c) Recall the isomorphism $\Psi : \text{Gal}(L/F) \simeq \widehat{\Delta(L/F)}$, given as $\Psi_h(t) = h(t^{1/n})/t^{1/n} \in \mu_n$. Define an action of G on $\widehat{\Delta(L/F)}$ as $(g \cdot \chi)(t) = g(\chi(g^{-1}t))$. Prove that the conjugation action of G on $\text{Gal}(L/F)$ (from the previous part) respects the map Ψ , in the sense that $\Phi_{ghg^{-1}} = g \cdot \Phi_h$ for every $g \in G$.
4. Let C_j denote the cyclic group of order j . Give an example of a Galois field extension $L/\mathbf{Q}$ for which $\text{Gal}(L/\mathbf{Q})$ is isomorphic to $(C_4 \times C_4) \rtimes C_2$, where C_2 acts on the $C_4 \times C_4$ by switching the factors and multiplying by -1 . **Hint:** Set $F = \mathbf{Q}(i)$, and observe that $\mu_4 \subseteq F$. Use Kummer theory to produce a $C_4 \times C_4$ extension L of F as $L = F(\alpha^{1/4}, \bar{\alpha}^{1/4})$, where $\bar{\alpha}$ is the complex conjugate of α . Now use the previous exercise.
5. Let G be a finite group. The group G is said to be *solvable* if there exists a chain of subgroups $1 = G_s \leq G_{s-1} \leq \cdots \leq G_1 \leq G_0 = G$ so that G_{j+1} is normal in G_j for each j , and the quotient group G_j/G_{j+1} is cyclic.
 - (a) Suppose G is solvable. Let $H \leq G$ be a subgroup. Prove that H is solvable.
 - (b) Suppose H is a normal subgroup of G , and G is solvable. Prove that the quotient group G/H is solvable.

- (c) Suppose H is a normal subgroup of a G . Suppose that H is solvable and G/H is solvable. Then G is solvable.

For this question, you do not need to turn anything in. Either write a proof yourself, or find a proof in a textbook or online and read it.