

Mathematics 100C Homework 4

Due: Wednesday May 3 2023

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Suppose F is a field of characteristic 0. Let L/F be a finite Galois extension with Galois group G , and $\gamma_1 \in L$. Let $g(x) \in F[x]$ be the irreducible polynomial for γ_1 , and let $g(x) = (x - \gamma_1) \cdots (x - \gamma_r) \in L[x]$ be its factorization in $L[x]$. (Recall that L splits every irreducible polynomial with a root in L .) Prove that G acts transitively on the roots γ_i , i.e., that $\{\gamma_1, \gamma_2, \dots, \gamma_r\}$ is the G -orbit of γ_1 . **Note:** We essentially did this in class. I am asking you to rewrite a proof of it, so that you are sure to remember this fact.
2. Suppose F is a field of characteristic 0. Let K/F be a finite extension of fields, and L/F be a finite Galois extension. Denote $\text{Hom}_F(K, L)$ the field embeddings $\sigma : K \rightarrow L$ that are the identity on F . Assume this Hom-set is non-empty; denote by $\iota : K \rightarrow L$ a fixed F -embedding. Let $G = \text{Gal}(L/F)$. Verify the following facts:
 - G acts transitively on $\text{Hom}_F(K, L)$ by $\tau \mapsto g \circ \tau$, $g \in G$ and $\tau \in \text{Hom}_F(K, L)$;
 - the stabilizer of ι in G is $\text{Gal}(L/\iota(K))$
 - $|\text{Hom}_F(K, L)| = [K : F]$.

Hint: Let γ be a primitive element for K/F , and $\gamma_1 = \iota(\gamma)$. Use the previous question.

3. Let the notations and assumptions be as in the previous question. For a primitive element γ for the extension K/F , prove that the norm $N_{K/F}(\gamma) = \prod_{\tau \in \text{Hom}_F(K, L)} \tau(\gamma)$. Here the left-hand side is an element of F , and the right-hand side is an element of L , and the claim is that the right-hand side is in $F \subseteq L$ and the two sides are equal. **Hint:** See the Exam 1 review problems for how to prove this equality.
4. Let the notation and assumptions be as in the previous questions. For an arbitrary element $\alpha \in K$, prove that the norm $N_{K/F}(\alpha) = \prod_{\tau \in \text{Hom}_F(K, L)} \tau(\alpha)$. **Hint:** Here is one way to proceed. For a general element $\alpha \in K$, we set $K_1 = F(\alpha) \subseteq K$. Prove that $N_{K/F}(\alpha) = N_{K_1/F}(\alpha)^{[K:K_1]}$, and that $\prod_{\tau \in \text{Hom}_F(K, L)} \tau(\alpha) = (\prod_{\sigma \in \text{Hom}_F(K_1, L)} \sigma(\alpha))^{[K:K_1]}$. Now deduce the claim from the previous problem, with K replaced by K_1 .