

Mathematics 100C Homework 3

Due: Wednesday April 26 2023

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Let p be a prime number. Let K be the splitting field of $x^4 - p$ in the complex numbers. By the first main theorem of Galois theory, K is Galois over $\mathbf{Q}$. (You may assume this.) Let $G = \text{Gal}(K/\mathbf{Q})$ be its Galois group. Prove that G is isomorphic to the dihedral group of order 8.
2. Fix a positive integer n . Suppose K is a subfield of the complex number that contains $\zeta_n = e^{2\pi i/n}$. Let $L = K(\gamma)$ be another subfield of $\mathbf{C}$, for which $\gamma \in L$ satisfies $\gamma^n \in K^\times$. Prove that L/K is Galois, with abelian Galois group. **Hint:** Set $\alpha = \gamma^n \in K^\times$. First observe that L is the splitting field of $x^n - \alpha \in K[x]$, so L is Galois, with some group G . Let $\mu_n = \{z \in \mathbf{C}^\times : z^n = 1\} = \{\zeta_n^a : a \in \mathbf{Z}\}$ be the n^{th} roots of unity. Define a map $\Phi : G \rightarrow \mu_n$ by $\Phi(g) = g(\gamma)/\gamma$. Prove that Φ is an injective group homomorphism.
3. Chapter 16, exercise 3.1
4. Chapter 16, exercise 3.2