

Mathematics 100C Homework 2

Due: Wednesday April 19 2023

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Suppose that $F = \mathbf{F}_p$, K is a field of order $q = p^r$, and $\alpha \in K^\times$ is a generator for this cyclic group. In class, we used that then $K = F(\alpha)$. The point of this exercise is to show that the converse is far from being true. To do this, prove the following.
 - (a) Suppose $\gamma \in K^\times$ is defined as $\gamma = \alpha^n$, with $n \in \mathbf{Z}$. Prove that $K = F(\gamma)$ if and only if, for all positive integers k strictly dividing r (i.e., $k|r$ and $k \neq r$), one has that n is not a multiple of $\frac{p^r-1}{p^k-1}$.
 - (b) Take for example $p = 3$, $r = 4$, and $n = 5$, so that $\gamma = \alpha^5$ in $K = \mathbf{F}_{81}$. Using part (a), observe that $K = F(\gamma)$ but that γ does not generate $K^\times$.
2. Chapter 15, exercise M.4. **Hint:** For part (d), observe that at least one of the equations $t^2 = 2, t^2 = 3, t^2 = 6$ has a solution in $\mathbf{F}_p$, and go from there.
3. Suppose F is a field of characteristic not 2, and K is a quadratic extension of F . Prove that there exists a nontrivial $\sigma \in \text{Aut}(K/F)$. Moreover, observe that if $\alpha \in K$, $\alpha \in F$ if and only if $\sigma(\alpha) = \alpha$.
4. Suppose F is a field of characteristic not 2 and K is a quadratic extension of F .
 - (a) Suppose $f(x) \in F[x]$ is an irreducible polynomial of odd degree. Prove that $f(x)$ remains irreducible in $K[x]$.
 - (b) Now suppose $f(x) \in F[x]$ is an irreducible polynomial of even degree $2n$. Prove that either one of the following two statements holds:
 - $f(x)$ remains irreducible in $K[x]$
 - $f(x) = g(x)h(x) \in K[x]$, with g, h irreducible of degree n in $K[x]$ and $h(x) = \sigma(g(x))$.

Hint Suppose $g(x) \in K[x]$ is an irreducible factor of $f(x)$ in $K[x]$ of degree at most $\deg(f)/2$. Set $h(x) = \sigma(g(x))$ and prove that $h(x)$ also divides $f(x)$ in $K[x]$.