

Mathematics 100C Final Exam Practice Problems

Instructions: The Final Exam is **Wednesday 14 June 2023, 11:30am-2:30pm (US Pacific time)**. It will be held in Warren Lecture Hall 2204, where we usually meet for lecture. No notes or calculators are allowed. This practice worksheet is to help you study for the Final Exam, which will cover the entire course. Specifically, the **Final will cover: finite fields; Galois theory (including Kummer theory); Noetherian rings, integer matrices, finitely generated abelian groups; representation theory**. Because you have already had much practice with finite fields and Galois theory, the practice worksheet focuses on the latter topics. However, the final will indeed cover the entire course, so do not forget to review the other topics. Note also that inclusion of a topic on this sheet does not guarantee that a similar problem will appear on the exam, nor does exclusion of a topic from this sheet imply that that topic will not be on the exam.

1. For a permutation $\sigma \in S_n$, let $f(\sigma)$ denote the number of fixed points of σ . Prove that $\sum_{\sigma \in S_n} f(\sigma)^2 = 2n!$. **Hint:** Let S_n act on $\mathbf{C}^n$ by permuting the coordinates, and let χ be the character of this representation. Compute $\langle \chi, \chi \rangle$ in two different ways.
2. Prove that the only 1-dimensional representations of the symmetric group S_n are the trivial representation and the sign representation.
3. Define explicitly the n irreducible representations of the group C_n .
4. Let C_n act on $\mathbf{C}^2$ by $k + n\mathbf{Z} \mapsto \begin{pmatrix} \cos(\theta_k) & -\sin(\theta_k) \\ \sin(\theta_k) & \cos(\theta_k) \end{pmatrix}$, where $\theta_k = 2\pi k/n$. Decompose this representation into irreducibles.
5. Let G be a finite group. Suppose the character table of G is given by an array A , so that the complex number A_{ij} is $\chi_i(g_j)$, where χ_i is the i^{th} irreducible representation of G and g_j is a representative of the j^{th} conjugacy class in G . (We arbitrarily order the irreps and the conjugacy classes.) Let c_j be the size of the j^{th} conjugacy class.
 - (a) Define a matrix T as $T_{ij} = \left(\frac{c_j}{|G|}\right)^{1/2} A_{ij}$. Prove that $TT^* = 1$.
 - (b) Deduce that the columns, and not just the rows, of the character table of G are orthogonal.
6. Suppose G is a finite group, that has a normal subgroup N of index 2. Suppose moreover that N is abelian. Prove that every irreducible representation of G has dimension either 1 or 2. **Hint:** Let V be the space of an irreducible representation of G . Restrict the action of G on V to N to find an eigenvector v for the action of N . If $\sigma \in G \setminus N$, let $w = \sigma(v)$ and prove that $\mathbf{C}v + \mathbf{C}w$ is G -stable.
7. Suppose N is a normal subgroup of G , ρ' an irreducible representation of G/N , and $\rho = \rho' \circ \pi$, where $\pi : G \rightarrow G/N$ is the quotient map. Prove that ρ is irreducible.
8. Suppose $\rho : G \rightarrow \text{GL}(V)$ is a representation, and $\dim(V) = d$.
 - (a) Let χ be the character of ρ . Prove $|\chi(g)| \leq d$, and $\chi(g) = d$ if and only if $\rho(g) = 1$.
 - (b) Explain, using the first part, how one can determine if a finite group is simple by looking at its character table.
9. Suppose $\rho : G \rightarrow \text{GL}(V)$ is an irreducible representation, and $\langle \cdot, \cdot \rangle_j$ for $j = 1, 2$ are G -invariant Hermitian forms on V . Suppose $\langle \cdot, \cdot \rangle_1$ is positive definite. Prove that there exists $\lambda \in \mathbf{C}$ so

- that $\langle v, w \rangle_2 = \lambda \langle v, w \rangle_1$ for all $v, w \in V$. **Hint:** For $v \in V$, consider the linear map $V \rightarrow \mathbf{C}$ given by $x \mapsto \langle v, x \rangle_2$. There exists a unique vector $A(v) \in V$ so that this linear map is $\langle A(v), x \rangle_1$. Prove that $v \mapsto A(v)$ is linear and G -equivariant, then apply Schur's lemma.
10. Suppose $\chi : G \rightarrow \mathbf{C}^\times$ is a homomorphism, and $\rho : G \rightarrow \text{GL}(V)$ a representation. Let $V(\chi)$ be the vector space V , but with G action given as $g \cdot v = \chi(g)\rho(g)(v)$. Prove that if ρ is irreducible, then so is the G -action on $V(\chi)$.
 11. Suppose G is a finite group. Recall that a *class function* $\varphi : G \rightarrow \mathbf{C}$ is a function that is constant on conjugacy classes.
 - (a) Suppose φ is a class function. Verify that $\varphi(g) = \sum_{\chi_i} \langle \chi_i, \varphi \rangle \chi_i$, where the sum is over the characters of the irreducible representations of G .
 - (b) Suppose now that $G = C_n$. Then a class function is any function $\varphi : \mathbf{Z}/n\mathbf{Z} \rightarrow \mathbf{C}$. Write out explicitly the identity $\varphi(g) = \sum_{\chi_i} \langle \chi_i, \varphi \rangle \chi_i$ in this case. The complex numbers $\langle \chi_i, \varphi \rangle$ in this case are called the Fourier coefficients of φ , and this identity can be thought of as part of the theory of the *finite Fourier transform*.
 12. Make the character tables for C_4 and K_4 , the cyclic group of order 4 and the Klein 4 group.
 13. Let $R = F[t]$ for a field F , and S a subring of R that contains F and is strictly bigger than F . Prove that R is a finitely generated S module.
 14. Suppose $v = (v_1, \dots, v_n) \in \mathbf{Z}^n$ is such that the v_j have no common divisor. Prove that v can be extended to a basis of $\mathbf{Z}^n$.
 15. Suppose $A \in M_n(\mathbf{Z})$ has $\det(A) = N$. Suppose $v \in N\mathbf{Z}^n$. Prove that there exists $w \in \mathbf{Z}^n$ so that $Aw = v$.
 16. Let V be a finite dimensional F vector space, and $T : V \rightarrow V$ a linear operator. Suppose $p(x) \in F[x]$ is irreducible of degree d , and $p(T) = 0$ on V . Prove that d divides $\dim(V)$.
 17. Let F be a field, V a finite dimensional F vector space, and $N : V \rightarrow V$ a linear map. One says that N is *nilpotent* if $N^k = 0$ for some positive integer k .
 - (a) Suppose the matrix of N is upper triangular, with 0's along the diagonal. Prove that N is nilpotent.
 - (b) Let V_r be the $r \times r$ matrix with 1's along the super diagonal and 0's elsewhere. That is, $V_{i,i+1} = 1$ for $i = 1, 2, \dots, r-1$ and the other matrix entries $V_{ij} = 0$. Prove that if N is nilpotent, then there is a basis of V so that in this basis, the matrix of N is block diagonal, with blocks of the form V_r for various integers r . **Hint:** Consider V as a module for $k[x]$, and use the structure theorem for modules over this ring that we proved.
 18. Let $G = C_{10} \times C_5 \times C_{100}$. How many elements $x \in G$ are there with $x^2 = 1$?
 19. Suppose $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z})$ has $\gcd(a, b, c, d) = u$ and $|\det(m)| = v = u^2 v_1$ for some positive integers u, v_1 . Let $P, Q \in \text{GL}_2(\mathbf{Z})$ be such that $m' := QmP^{-1} = \text{diag}(a_1, a_2)$ with $a_1, a_2 > 0$ and $a_1 | a_2$. What are a_1 and a_2 ?
 20. Let G be the abelian group generated by elements x, y with $x^{20} = 1$, $y^{30} = 1$, and $x^2 = y^3$.
 - (a) Write a presentation matrix for the group G .
 - (b) Describe G explicitly as a product of cyclic groups.