Final Exam: Math 100B, Winter 2023

You have 3 hours.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name ____

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ____

There are 10 problems; each problem is worth 20 points.

Good luck!

Problem 1. *(20 points) Answer "True" if the statement is true, and "False" if the statement is false. For this question, you do not need to show work or provide an explanation.*

1. *(5 points) There exists a principal ideal domain of characteristic* 2.
   TRUE. For example $\mathbf{F}_2[x]$.

2. *(5 points) There exists a ring* R *of finite size and characteristic* 0.
   FALSE. If R *has finite size, then the map* $\mathbf{Z} \to R$ *must have some kernel.*

3. *(5 points) A ring* R *is an integral domain if and only if there exists a field* F *so that* R *is isomorphic to a subring of* F.
   TRUE. A subring of a field is always an integral domain. For the converse, apply the fraction field construction.

4. *(5 points) Suppose*
   - F *is a field,*
   - V_1, V_2 *are finite-dimensional vector spaces over* F *of the same dimension, and*
   - $T : V_1 \to V_2$ *is an injective linear map.*

   Then T *is necessarily also surjective.*
   TRUE. The image $T(V_1)$ *is a subspace of* V_2 *with dimension* $\dim(V_1) = \dim(V_2)$, *so it is equal to* V_2.

Problem 2. *(20 points) Prove that the group of units of the ring* $\mathbf{Z}[\sqrt{5}]$ *is infinite.*

Proof. Set $S = \mathbf{Z}[\sqrt{5}]$ and $u = \sqrt{5} + 2$. Then $u \in S^\times$ because $(\sqrt{5} + 2)(\sqrt{5} - 2) = 5 - 4 = 1$. Because $u > 1$ then powers u^n for $n \in \mathbf{Z}$ are all distinct, and all in $S^\times$, so S has infinitely many units. □

Problem 3. *(20 points) Prove that every nonzero ideal* I *of the ring* $\mathbf{Z}[\sqrt{5}]$ *contains a nonzero integer.*

Proof. Suppose $a + b\sqrt{5} \in I$ with at least one of a, b nonzero. Then $(a + b\sqrt{5})(a - b\sqrt{5}) = a^2 - 5b^2$ is in I. But this is an integer, and it is nonzero because $a^2 = 5b^2$ contradicts unique factorization in $\mathbf{Z}$ (the exponent of 5 on the LHS is even but the exponent is odd on the RHS.) □

Problem 4. *(20 points) Suppose V is a 10-dimensional vector space over the rational numbers $\mathbf{Q}$, and $T : V \to V$ is a linear map. Set $S = T \circ T \circ T \circ T \circ T$, so that S is a linear map $V \to V$. Suppose moreover that $\det(S) = 1$. What are all the possibilities for $\det(T)$? Be sure to prove your claims.*

Proof. We must have $\det(T) = 1$. Indeed, $\det(T)$ is a rational number, and $\det(T)^5 = \det(S) = 1$; the only rational number whose $5th$ power is 1 is 1, so $\det(T) = 1$. Moreover, this possibility is realized by taking T to be the identity map. □

Problem 5. *(20 points) Find elements u and v in the ring $\mathbf{Z}[x]$ so that $\mathbf{Z}[x]/(u, v)$ is a field with 9 elements. Be sure to prove your claims.*

Proof. Let $u = 3$ and $v = x^2 + 1$. Then $\mathbf{Z}[x]/(u, v) = \mathbf{F}_3[x]/(x^2 + 1)$. Because $x^2 + 1$ has degree 2, this ring has size $3^2 = 9$. Because $x^2 + 1$ does not have a root in $\mathbf{F}_3$, it is irreducible in $\mathbf{F}_3[x]$, so $\mathbf{F}_3[x]/(x^2 + 1)$ is a field. □

Problem 6. *(20 points) Consider the ring homomorphism* $\varphi : \mathbf{Q}[x, y] \to \mathbf{Q}[t]$ *given by* $\varphi(f(x, y)) = f(t, t^2 - 1)$*. Find an element* $g(x, y) \in \mathbf{Q}[x, y]$ *so that* $\ker(\varphi) = (g(x, y))$*. Be sure to prove your results.*

Proof. Set $g(x, y) = y - x^2 + 1$. Clearly $g(x, y) \in \ker(\varphi)$. To see that it generates the kernel, suppose $f(x, y) \in \ker(\varphi)$. Because g is monic in y, we can apply division with remainder to obtain $f(x, y) = q(x, y)g(x, y) + r(x)$. The remainder only depends upon x because $g(x, y)$ has degree 1 in y. Thus $0 = \varphi(r(x)) = r(t)$, so $r(x) = 0$ and thus g divides f in $\mathbf{Q}[x, y]$ as desired. □

Problem 7. *(20 points) Let* $R = \mathbf{Z}[x]/(x^3)$*. Does there exist integral domains* $S_1, S_2, \ldots, S_n$ *and an injective ring homomorphism* $\varphi : R \to S_1 \times S_2 \times \cdots \times S_n$*? Be sure to prove your claims.*

Proof. The ring R has a nonzero nilpotent element, namely, the image of x in R. On the other hand, if $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ is nilpotent, then all s_j are 0 because the S_j are integral domains. Thus if $\varphi : R \to S_1 \times \cdots \times S_n$ is a ring homomorphism, $\varphi(x) = 0$ so φ is not injective. □

Problem 8. *(20 points) Give an example of subfields E_1 and E_2 of the complex numbers* $\mathbf{C}$ *so that*

- *the intersection $E_1 \cap E_2$ in* $\mathbf{C}$ *is* $\mathbf{Q}$*;*
- *E_1 and E_2 are finite extensions of* $\mathbf{Q}$ *of degree at least 2;*
- *E_1 is isomorphic to E_2 as fields.*

Be sure to prove your claims.

Proof. Let $\alpha_1 = 2^{1/3}$ and $\alpha_2 = \zeta_3 2^{1/3}$, where $\zeta_3 = e^{2\pi i/3}$. Set $E_1 = \mathbf{Q}(\alpha_1)$ and $E_2 = \mathbf{Q}(\alpha_2)$. Now $x^3 - 2 \in \mathbf{Q}[x]$ is irreducible by Eisenstein's criterion. As α_1, α_2 are roots of $x^3 - 2$, we have ring homomorphisms $\phi_1 : \mathbf{Q}[x]/(x^3 - 2) \to E_1$ and $\phi_2 : \mathbf{Q}[x]/(x^3 - 2) \to E_2$. These ϕ_j are isomorphisms, because $x^3 - 2$ is irreducible, so the degree $[E_j : \mathbf{Q}] = 3$. Consequently, $\phi_2 \circ \phi_1^{-1} : E_1 \to E_2$ is an isomorphism of fields.

To see that $E_1 \cap E_2 = \mathbf{Q}$, let E be the intersection. Then E is a field, so the degree $[E : \mathbf{Q}]$ divides $[E_2 : \mathbf{Q}] = 3$. The degree cannot be 3, because then $E_2 = E_1 \cap E_2$ so $E_2 \subseteq E_1$, which contradicts the fact that $E_1 \subseteq \mathbf{R}$ but E_2 is not contained in $\mathbf{R}$. Hence $[E : \mathbf{Q}] = 1$ so $E = \mathbf{Q}$. $\square$

Problem 9. *(20 points) Give an example of an ideal J in $\mathbf{Z}[x]$ so that $\mathbf{Z}[x]/J$ is isomorphic to $\mathbf{F}_5 \times \mathbf{Z}[\sqrt{2}]$. Be sure to prove your claims.*

Proof. Set $M = (5, x)$ and $I = (x^2 - 2)$, ideals of $\mathbf{Z}[x]$. Clearly $\mathbf{Z}[x]/M \simeq \mathbf{F}_5$ and $\mathbf{Z}[x]/I \simeq \mathbf{Z}[\sqrt{2}]$. Now $M + I = (1)$, as $5 - 2x^2 + 2(x^2 - 2) = 1$. Thus if $J = M \cap I = MI$, then CRT applies to give $\mathbf{Z}[x]/J = \mathbf{F}_5 \times \mathbf{Z}[\sqrt{2}]$. □

Problem 10. *(20 points) Let $\varphi : \mathbf{Q}[x, y] \to \mathbf{C}$ denote the ring homomorphism with $\varphi(x) = \sqrt{3}$ and $\varphi(y) = 2^{1/3}$. Prove that* $\ker(\varphi) = (x^2 - 3, y^3 - 2)$.

Hint: *Let M be the ideal $(x^2 - 3, y^3 - 2)$. Prove that any monomial $x^m y^n$ in $\mathbf{Q}[x, y]$ is equivalent modulo M to an expression $\alpha x^i y^j$ with $0 \leq i \leq 1$, $0 \leq j \leq 2$, and $\alpha \in \mathbf{Q}$. Consequently, the dimension of $\mathbf{Q}[x, y]/M$ as a vector space over $\mathbf{Q}$ is at most 6.*

Proof. If $m \geq 2$, then $x^m \equiv 3x^{m-2} \pmod{M}$ and if $n \geq 3$, then $y^n \equiv 2y^{n-3} \pmod{M}$. Repeating, it is clear that the images of $x^i y^j$ with $0 \leq i \leq 1$, $0 \leq j \leq 2$ span $\mathbf{Q}[x, y]/M$ as a $\mathbf{Q}$ vector space, so $\dim_{\mathbf{Q}} \mathbf{Q}[x, y]/M \leq 6$.

Set $E = \mathbf{Q}(3^{1/2}, 2^{1/3}) \subseteq \mathbf{C}$. Because $[\mathbf{Q}(\sqrt{3}) : \mathbf{Q}] = 2$ and $[\mathbf{Q}(2^{1/3}) : \mathbf{Q}] = 3$, and these degrees are relatively prime, $[E : \mathbf{Q}] = 6$.

Now let $K = \ker(\varphi)$. Clearly $K \supseteq M$, so we obtain an induced map

$$\mathbf{Q}[x, y]/M \to \mathbf{Q}[x, y]/K \to E.$$

This is a linear map of $\mathbf{Q}$-vector spaces. The left-hand space has dimension at most 6, while the right hand space has dimension exactly 6, and the map is surjective. Thus $\mathbf{Q}[x, y]/M$ has dimension exactly 6 and the map is an isomorphism. Consequently $M = K$ as desired. □

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