Final Exam: Math 100B, Winter 2023

You have 3 hours.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name ____

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ____

There are 10 problems; each problem is worth 20 points.

Good luck!

Problem 1. *(20 points) Answer "True" if the statement is true, and "False" if the statement is false. For this question, you do not need to show work or provide an explanation.*

1. *(5 points) There exists a principal ideal domain of characteristic* 2.

2. *(5 points) There exists a ring R of finite size and characteristic* 0.

3. *(5 points) A ring R is an integral domain if and only if there exists a field F so that R is isomorphic to a subring of F.*

4. *(5 points) Suppose*
   - *F is a field,*
   - *V_1, V_2 are finite-dimensional vector spaces over F of the same dimension, and*
   - *$T : V_1 \to V_2$ is an injective linear map.*

   Then T is necessarily also surjective.

Problem 2. *(20 points) Prove that the group of units of the ring* $\mathbf{Z}[\sqrt{5}]$ *is infinite.*

Problem 3. *(20 points) Prove that every nonzero ideal* I *of the ring* $\mathbf{Z}[\sqrt{5}]$ *contains a nonzero integer.*

Problem 4. *(20 points) Suppose V is a 10-dimensional vector space over the rational numbers $\mathbf{Q}$, and $T : V \to V$ is a linear map. Set $S = T \circ T \circ T \circ T \circ T$, so that S is a linear map $V \to V$. Suppose moreover that $\det(S) = 1$. What are all the possibilities for $\det(T)$? Be sure to prove your claims.*

Problem 5. *(20 points) Find elements u and v in the ring $\mathbf{Z}[x]$ so that $\mathbf{Z}[x]/(u, v)$ is a field with 9 elements. Be sure to prove your claims.*

Problem 6. *(20 points) Consider the ring homomorphism* $\varphi : \mathbf{Q}[x, y] \to \mathbf{Q}[t]$ *given by* $\varphi(f(x, y)) = f(t, t^2 - 1)$. *Find an element* $g(x, y) \in \mathbf{Q}[x, y]$ *so that* $\ker(\varphi) = (g(x, y))$. *Be sure to prove your results.*

Problem 7. *(20 points) Let* $R = \mathbf{Z}[x]/(x^3)$. *Does there exist integral domains* $S_1, S_2, \ldots, S_n$ *and an injective ring homomorphism* $\varphi : R \to S_1 \times S_2 \times \cdots \times S_n$*? Be sure to prove your claims.*

Problem 8. *(20 points) Give an example of subfields E_1 and E_2 of the complex numbers $\mathbf{C}$ so that*

- *the intersection $E_1 \cap E_2$ in $\mathbf{C}$ is $\mathbf{Q}$;*
- *E_1 and E_2 are finite extensions of $\mathbf{Q}$ of degree at least 2;*
- *E_1 is isomorphic to E_2 as fields.*

Be sure to prove your claims.

Problem 9. *(20 points) Give an example of an ideal J in $\mathbf{Z}[x]$ so that $\mathbf{Z}[x]/J$ is isomorphic to $\mathbf{F}_5 \times \mathbf{Z}[\sqrt{2}]$. Be sure to prove your claims.*

Problem 10. *(20 points) Let* $\varphi : \mathbf{Q}[x, y] \to \mathbf{C}$ *denote the ring homomorphism with* $\varphi(x) = \sqrt{3}$ *and* $\varphi(y) = 2^{1/3}$*. Prove that* $\ker(\varphi)$ *is the ideal* $(x^2 - 3, y^3 - 2)$ *of* $\mathbf{Q}[x, y]$*.*

Hint*: Let* M *be the ideal* $(x^2 - 3, y^3 - 2)$*. Prove that any monomial* $x^m y^n$ *in* $\mathbf{Q}[x, y]$ *is equivalent modulo* M *to an expression* $\alpha x^i y^j$ *with* $0 \leq i \leq 1$, $0 \leq j \leq 2$, *and* $\alpha \in \mathbf{Q}$*. Consequently, the dimension of* $\mathbf{Q}[x, y]/M$ *as a vector space over* $\mathbf{Q}$ *is at most* 6*. This fact might help you prove that* $M = \ker(\varphi)$*.*

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