Exam 2: Math 100B, Winter 2023

You have 50 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name ____

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ____

There are 5 problems; each problem is worth 20 points.

Good luck!

Problem 1. *(20 points) This problem has two parts.*

1. *(10 points) Suppose F is a field, and $f(x) \in F[x]$ factorizes into irreducibles as $f(x) = p_1(x)^{e_1} \cdots p_m(x)^{e_m}$. That is, each $p_j(x) \in F[x]$ is irreducible, each e_j is a positive integer, and $p_i(x)$ shares no common factor with $p_j(x)$ if $i \neq j$. Set $R_j = F[x]/(p_j(x)^{e_j})$. Prove that $F[x]/(f(x))$ is isomorphic to $R_1 \times \cdots \times R_m$.*

2. *(10 points) Give an example of a polynomial $f(x) \in \mathbf{R}[x]$ such that $\mathbf{R}[x]/(f(x))$ is isomorphic to $\mathbf{C} \times \mathbf{C}$. Be sure to prove your claims. (For this problem, you may use the result of the first part, regardless of whether you proved it.)*

Problem 2. *(20 points) Give an example of an integral domain R, and a nonzero element $a \in R$, such that a is irreducible but not prime. Be sure to prove your claims.*

Problem 3. *(20 points) Give an example of a unique factorization domain R, and two nonzero elements $a, b \in R$ such that*

- *the greatest common divisor of a and b is 1,*
- *the element 1 is **not** expressible in the form $ra + sb$, $r, s \in R$.*

Be sure to prove your claims.

Problem 4. *(20 points) Suppose $f(x) \in \mathbf{Z}[x]$ and $g(x) \in \mathbf{Z}[x]$ are monic polynomials of positive degree. Let $I = (f(x), g(x))$ be the ideal in $\mathbf{Z}[x]$ generated by $f(x)$ and $g(x)$, and set $R = \mathbf{Z}[x]/I$. Recall that we say that $f(x)$ and $g(x)$ have a common factor in $\mathbf{Q}[x]$ if there exists a nonconstant polynomial $h(x) \in \mathbf{Q}[x]$ such that $h(x)$ divides both $f(x)$ and $g(x)$ in $\mathbf{Q}[x]$.*

1. *(10 points) Suppose $f(x)$ and $g(x)$ have a common factor in $\mathbf{Q}[x]$. Prove that, in this case, R has infinitely many elements.*

2. *(10 points) Suppose $f(x)$ and $g(x)$ do not have a common factor in $\mathbf{Q}[x]$. Now prove that R has finitely many elements.*

Problem 5. *(20 points) Suppose p is a prime number, and a, b are integers such that $a^2 + b^2 = p$. Prove that the quotient ring $\mathbf{Z}[i]/(a + ib)$ is a field.*

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