Exam 1: Math 100B, Winter 2023

You have 50 minutes.

You are not permitted to use calculators, books, or notes.

YOU MUST SHOW ALL YOUR WORK TO RECEIVE CREDIT.

Name ____

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ____

There are 5 problems; each problem is worth 20 points.

Good luck!

Problem 1. *(20 points) Suppose p is a prime number and set $R = \mathbf{Z}/p\mathbf{Z}$. Determine the size of the unit group of the polynomial ring $R[x]$ explicitly in terms of p. Be sure to prove your answer.*

Problem 2. *(20 points) The two parts of this question are not related to one another. For both parts, be sure to justify your answer.*

1. *(10 points) Give an example of a ring R that has finitely many elements, and contains at least 3 distinct ideals.*

2. *(10 points) Give an example of a ring of characteristic 0 that is not a field.*

Problem 3. *(20 points) Suppose R is a ring in which every ideal is principal. Let $a \in R$ be fixed, and set $I = (x - a) \subseteq R[x]$ the principal ideal of $R[x]$ generated by $x - a$. If J is an ideal of $R[x]$ that contains I, prove that J can be generated by two elements, i.e., that there exists $r_1(x), r_2(x) \in R[x]$ so that $J = (r_1(x), r_2(x))$.*

Problem 4. *(20 points) Let* $R = \mathbf{Z}[i] = \{a + bi : a, b \in \mathbf{Z}\}$ *be the ring of Gaussian integers. Suppose* $I \subseteq R$ *is a nonzero ideal.*

1. *(10 points) Prove that there exists a nonzero integer* $n \in I$*.*

2. *(10 points) Prove that the quotient abelian group* R/I *has finite size.* For this question, you may assume that there exists a nonzero integer $n \in I$, regardless of whether you proved that in part (1).

Problem 5. *(20 points) Consider the map* $\varphi : \mathbf{Q}[x, y] \to \mathbf{Q}[t]$ *given by* $\varphi(f(x, y)) = f(t, t^3 + t^2 + 3)$*. Find a polynomial* $g(x, y) \in \mathbf{Q}[x, y]$ *so that* $\ker(\varphi)$ *is the principal ideal generated by* $g(x, y)$*. Be sure to prove your result.*

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