Mathematics 100B Homework 9
Due: Wednesday March 15 2023

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Let V be a finite-dimensional F vector space. Let $B = (b_1, \ldots, b_n)$ be a basis of V. Suppose $T : V \to V$ is a linear map, and $Tb_j = \sum_i a_{ij} b_i$ with the $a_{ij} \in F$. The matrix of T with respect to the basis B is defined to be that element m of $M_n(F)$ with (i, j) component equal to a_{ij}. Suppose T_1, T_2 are linear maps from $V \to V$ with associated matrices m_1, m_2 with respect to the same basis B. Prove that the matrix of $T_1 \circ T_2$ is the matrix product $m_1 m_2$.

2. Let V be a finite-dimensional F vector space, and $B = (b_1, \ldots, b_n)$ and $B' = (b'_1, \ldots, b'_n)$ bases of V. Let $S : V \to V$ be the linear map defined as $S(b_j) = b'_j$ for $j = 1, 2, \ldots, n$. Suppose $T : V \to V$ is a linear map, with matrix $m_B(T)$ with respect to B. Let $m_{B'}(T)$ be the matrix of T with respect to B', and let $m_B(S)$ be the matrix of S with respect to B. Prove that $m_{B'}(T) = m_B(S)^{-1} m_B(T) m_B(S)$. **Hint**: Suppose $Tb'_j = \sum_i m'_{ij} b'_i$. Applying S^{-1} to both sides, one obtains $S^{-1}Tb'_j = \sum_i m'_{ij} b_i$. Consequently $S^{-1}TSb_j = \sum_i m'_{ij} b_i$.

3. Let $m \in M_n(F)$ be an $n \times n$ matrix with coefficients in F. Define the trace of m as the sum of the diagonal entries of m: $\mathrm{tr}(m) = \sum_i m_{ii}$. Prove that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for all $X, Y \in M_n(F)$. Now let $T : V \to V$ be a linear transformation, where V is an n-dimensional F vector space. Define the trace of T as $\mathrm{tr}(T) = \mathrm{tr}(m_B(T))$, the trace of the matrix of T with respect to any basis B of V. Prove that this quantity is independent of the basis B chosen, and thus is well-defined.

4. Suppose $m \in M_n(F)$ for a field F. The determinant of m can be defined just as in Section 1.4 of the text. It has the property that $\det(AB) = \det(A)\det(B)$ and that A is an invertible matrix if and only if $\det(A) \neq 0$ in F. Read section 1.4 in the text, or some other book, to verify that these properties hold. You do not need to turn in anything for this problem.

5. Suppose $T : V \to V$ is a linear transformation of a finite-dimensional F vector space V. Let B be a basis of V, $m_B(T)$ the matrix of T with respect to B. Define $\det(T) := \det(m_B(T))$. Prove that this is independent of the basis chosen, and thus well-defined.