

## Mathematics 100B Homework 6

### Due: Wednesday February 22 2023

**Instructions:** Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Prove that the elements  $a = 6$  and  $b = 2 + 2\sqrt{-5}$  in  $R = \mathbf{Z}[\sqrt{-5}]$  do not have a greatest common divisor. **Hint:** For  $x = u + \sqrt{-5}v \in R$ , set  $N(x) = u^2 + 5v^2$ . Observe that if  $x$  divides  $y$  in  $R$ , then  $N(x)$  divides  $N(y)$  in  $\mathbf{Z}$ . Consequently, if  $e$  is a common divisor of  $a, b$ , one sees that  $N(e)$  divides 12. On the other hand, if  $d$  is a greatest common divisor of  $a, b$ , then  $4 = N(2)$  divides  $N(d)$  and  $6 = N(1 + \sqrt{-5})$  divides  $N(d)$ , so 12 divides  $N(d)$ . Conclude that if  $d$  is a gcd of  $a, b$ , then  $N(d) = 12$ . Use this, together with the fact that 2 divides  $d$ , to see that  $d$  cannot exist.
2. Chapter 12, Exercise 2.1
3. How many roots does the polynomial  $x^2 - 1$  have, modulo 8?
4. Chapter 12, Exercise 2.6
5. Chapter 12, Exercise 3.2