Mathematics 100B Homework 4
Due: Wednesday February 8 2023

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. In class, we discussed the fact that $\mathbf{C}[t]/(t^2-1)$ has 4 ideals. Prove that this ring is isomorphic to the product ring $\mathbf{C} \times \mathbf{C}$.

2. In a previous homework exercise, you checked that $\mathbf{Z}[x]/(6, 2x-1)$ is isomorphic to $\mathbf{Z}/3\mathbf{Z}$. Prove that 7 is not in the ideal $(6, 2x-1)$ of $\mathbf{Z}[x]$. More generally, determine exactly the set of integers n such that $n \in (6, 2x-1)$.

3. Let $\mathbf{F}_7 = \mathbf{Z}/7\mathbf{Z}$ be the field with 7 elements. Prove that $\mathbf{Z}[x]/(7, x^2-2)$ is isomorphic to $\mathbf{F}_7 \times \mathbf{F}_7$.

4. In class, we defined the product of two rings R_1, R_2. If $R_1, \ldots, R_n$ are rings, define for yourself the product $R_1 \times \cdots \times R_n$, and check that it is a ring. You do not need to turn in anything for this question.

5. Prove the following generalization of the Chinese Remainder Theorem: Let R be a ring, $I_1, \ldots, I_n$ ideals of R, with $I_j + I_k = R$ for every pair j, k with $j \neq k$. Prove that the canonical map from $R/(I_1 \cdots I_n)$ to $R/I_1 \times R/I_2 \times \cdots \times R/I_n$ is an isomorphism. **Hint**: Prove that $(I_1 \cdots I_{n-1}) + I_n = R$ and then induct on n.

6. Chapter 11, Exercise 6.7