Mathematics 100B Exam 2 Practice Problems

Instructions: Exam 2 is Friday February 24. It will be an in class exam, 50 minutes long. No notes or calculators are allowed. This practice worksheet is to help you study for Exam 2. Anything discussed up to and including the material in lecture on Friday February 17 is fair game for the exam. Note that inclusion of a topic on this sheet does not guarantee that a similar problem will appear on the exam, nor does exclusion of a topic from this sheet imply that that topic will not be on the exam. Exam 2 will consist of 5 problems, of a range of difficulty, that mirrors the difficulty of problems on this worksheet.

1. Let R, S be rings. Prove that a subset U of $R \times S$ is an ideal if and only if $U = I \times J = \{(x, y) : x \in I, y \in J\}$, where I is an ideal of R and J is an ideal of S.

2. Suppose p is a prime number. Give an example of a ring with exactly one maximal ideal, and p^3 elements.

3. Suppose F is a field. Prove that $F[x]$ contains infinitely many monic irreducible polynomials. (**Hint**: Suppose that there were only finitely many, $f_1(x), f_2(x), \ldots, f_m(x)$. Set $g(x) = f_1(x)f_2(x)\cdots f_m(x) + 1$. Consider the factorization of g.) Deduce that if p is a prime, there are infinitely many positive integers n so that there exists a field with p^n elements. (We will later prove that for every n, there is a field with p^n elements, and that this field is unique up to isomorphism.)

4. Let $f(x) = x^3 + x^2 + x + 1 \in \mathbf{R}[x]$. Factor f into irreducibles in $\mathbf{R}[x]$. **Hint**: Observe that $f(-1) = 0$.

5. Suppose $f_1(x) = x^2 + b_1x + c_1 \in \mathbf{R}[x]$ and $f_2(x) = x^2 + b_2x + c_2 \in \mathbf{R}[x]$. Suppose moreover that $b_1^2 - 4c_1 < 0$ and $b_2^2 - 4c_2 < 0$. Prove that $\mathbf{R}[x]/(f_1(x))$ is isomorphic to $\mathbf{R}[x]/(f_2(x))$.

6. Suppose $f(x) \in \mathbf{Z}[x]$ is a monic polynomial, that is irreducible in $\mathbf{Q}[x]$. Set $R = \mathbf{Z}[x]/(f(x))$ and $F = \mathbf{Q}[x]/(f(x))$. Prove that R is an integral domain, and that F can be identified with the fraction field of R.

7. Suppose F is a field, $f(x) \in F[x]$ and $f(x) = p_1(x)^{e_1} \cdots p_m(x)^{e_m}$ is the factorization of f into irreducibles. That is, each p_j is irreducible, and p_j is not an associate of p_k if $j \neq k$. Set $R_j = F[x]/(p_j(x)^{e_j})$. Prove that $F[x]/(f(x))$ is isomorphic to $R_1 \times \cdots \times R_m$.

8. Prove that the ring $\mathbf{Z}[x]/(7, x^3 + x^2 + x + 1)$ is isomorphic to $\mathbf{F}_7 \times E$ where E is a field with 49 elements.

9. Prove that $\mathbf{Z}[\sqrt{-3}]$ is not a UFD. **Hint**: Consider the equality $2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$.

10. Suppose $F \subseteq E$ are fields, and $\gamma \in E$ is such that there exists a nonzero polynomial $f(x) \in F[x]$ so that $f(\gamma) = 0$. Let $R = F[\gamma] \subseteq E$. Prove that R is a field.