Mathematics 100B Exam 1 Practice Problems

Instructions: Exam 1 is Friday January 27. It will be an in class exam, 50 minutes long. No notes or calculators are allowed. This practice worksheet is to help you study for Exam 1. Anything discussed up to and including the material in lecture on Friday January 20 is fair game for the exam. Note that inclusion of a topic on this sheet does not guarantee that a similar problem will appear on the exam, nor does exclusion of a topic from this sheet imply that that topic will not be on the exam. Exam 1 will consist of 5 problems, of a range of difficulty, that mirrors the difficulty of problems on this worksheet.

1. How many units are there in $\mathbf{Z}/15\mathbf{Z}$?

2. Let $R = \mathbf{Z}/15\mathbf{Z}$.

   (a) Give an example of two nonzero polynomials f, g in $R[x]$ of positive degree such that the product $f(x)g(x) = 0$ in $R[x]$.

   (b) Suppose $f(x) \in R[x]$ has leading coefficient a unit in R, i.e., $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \in R^\times$. Prove that if $g(x) \in R[x]$ and $f(x)g(x) = 0$, then $g(x) = 0$.

3. Let $R = \mathbf{Z}[\sqrt{2}]$.

   (a) Prove that the ring R has infinitely many units. **Hint**: First prove that $3 - 2\sqrt{2}$ is a unit.

   (b) Prove that every element of R is an algebraic number.

4. Let $R = \mathbf{Z}[\sqrt{2}]$ and let S be the subset of $M_2(\mathbf{Z})$ the (2×2 matrices with integer entries) consisting of matrices of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$. That is,

$$S = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbf{Z} \right\}.$$

   Prove that S is a ring with the usual matrix addition and multiplication, and that R is isomorphic to S.

5. Recall that an element x of a ring R is said to be *nilpotent* if there exists a positive integer N so that $x^N = 0$ in R. Let R be a ring, and I the subset of nilpotent elements of R. Prove that I is an ideal of R.

6. Find generators for the kernel of the map $\varphi : \mathbf{Z}[x] \to \mathbf{R}$ defined as $\varphi(f(x)) = f(\sqrt{3})$. (For this problem, you may use, without proving, that $\sqrt{3}$ is irrational.)

7. Suppose $x \in \mathbf{C}$ is an algebraic number. Prove that $2x$ is also an algebraic number.

8. Suppose that S is a ring with 35 elements. Either:

   (a) Give an example of such an S with characteristic 3, OR

   (b) prove that S cannot have characteristic 3.

9. Let $R = \mathbf{Z}/5\mathbf{Z}$.

(a) Give an example a ring homomorphism $\varphi : R[x,y] \to R[x,y]$ that is injective but not surjective.

(b) Give an example of a ring homomorphism $\psi : R[x,y] \to R[x,y]$ that is surjective but not the identity.

10. Prove that there does not exists a ring homomorphism from $\mathbf{Z}[\sqrt{2}]$ to $\mathbf{Z}[\sqrt{3}]$. (For this problem, you may use, without proving, that $\sqrt{3}$ is irrational.)