Mathematics 100A Homework 9  
Due: Wednesday 2 December 2020

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Please feel free to reach out to me or the TA if you have any questions.

1. Chapter 6, exercise 10.2

Proof. Suppose that \( s_1, s_2 \) are two elements of \( S \), and suppose that \( s_1 \) is contained in the distinct sets \( g_1 U, g_2 U, \ldots g_n U \). Because the action of \( G \) on \( S \) is assumed transitive, there exists \( h \in G \) with \( hs_1 = s_2 \). Thus \( s_2 \) is contained in \( h g_1 U, h g_2 U, \ldots h g_n U \). Moreover, it is clear that these \( m \) subsets of \( S \) are distinct, because the \( g_1 U, g_2 U, \ldots g_n U \) are distinct. It follows that if \( s_1 \) is contained in \( m \) distinct sets then so is \( s_2 \) for any pair of elements \( s_1, s_2 \in S \). The claim follows. \( \square \)

2. Denote by \( S_n \) the symmetric group of permutations of \( \{1, 2, \ldots, n\} \). Define a map \( S_n \times \mathbb{Z}^n \to \mathbb{Z}^n \) as

\[
(p, (a_1, a_2, \ldots, a_n)) \mapsto (a_{p^{-1}(1)}, a_{p^{-1}(2)}, \ldots, a_{p^{-1}(n)}).
\]

Here \( p \in S_n \) and \( a_1, a_2, \ldots, a_n \in \mathbb{Z} \). Prove that this map defines an action of \( S_n \) on \( \mathbb{Z}^n \) by group homomorphisms.

Proof. Suppose \( p, q \in S_n \). Define \( b_j = a_q^{-1}(j) \). Then \( p \ast q \ast (a_1, \ldots, a_n) = p \ast (b_1, \ldots, b_n) = (b_{p^{-1}(1)}, \ldots, b_{p^{-1}(n)}) \). But now \( b_{p^{-1}(j)} = a_{q^{-1}p^{-1}(j)} = a_{(pq)^{-1}(j)} \). Thus \( p \ast q \ast (a_1, \ldots, a_n) = (pq) \ast (a_1, \ldots, a_n) \) so the formula given in the problem defines an action. It is clear that this action is by group homomorphisms:

\[
p \ast ((a_1, \ldots, a_n) + (a_1', \ldots, a_n')) = p \ast (a_1 + a_1', \ldots, a_n + a_n')
= (a_{p^{-1}(1)} + a_{p^{-1}(1)}', \ldots, a_{p^{-1}(n)} + a_{p^{-1}(n)}')
= p \ast (a_1, \ldots, a_n) + p \ast (a_1', \ldots, a_n').
\]

\( \square \)

3. Suppose \( A \) is an abelian group, and a group \( G \) acts on \( A \) by group homomorphisms. Denote \( A^G = \{a \in A : g \ast a = a \forall g \in G \} \).

(a) Prove that \( A^G \) is contained in the center of the semidirect product \( A \rtimes_\phi G \), where \( \phi : G \to \text{Aut}(A) \) is the homomorphism derived from the action of \( G \) on \( A \).

(b) Let \( Z(G) \) denote the center of \( G \), and denote by \( H \) the subgroup of \( Z(G) \) defined as

\[
H = \{z \in Z(G) : z \ast a = a \forall a \in A \}.
\]

Prove that the center of \( A \rtimes_\phi G \) is the direct product \( A^G \times H \).
Proof. Of course the second statement implies the first, so we prove the second. Because $A$ and $G$ generate $A \rtimes \phi G$, an element is in the center of the semidirect product if and only if it commutes with all elements of $A$ and all elements of $G$.

Now, we compute $(a,1)(b,g)(a,1)^{-1} = (ab,g)(a^{-1},1) = (abg \ast (a^{-1}),g) = (bag \ast (a^{-1}),g)$ because $A$ is assumed abelian. This equals $(b,g)$ if and only if $ag \ast (a^{-1}) = 1$, equivalently $a = g \ast a$. Thus $(b,g)$ commutes with every element of $A$ if and only if $g \ast a = a$ for all $a \in A$.

Similarly, we compute $(1,h)(b,g)(1,h)^{-1} = (h \ast b,hg)(1,h^{-1}) = (h \ast b,hgh^{-1})$. This equals $(b,g)$ if and only if $hgh^{-1} = g$ and $h \ast b = b$. Thus $(b,g)$ commutes with every element of $G$ if and only if $g \in Z(G)$ and $b \in A^G$.

Combining the two conditions, we find $(b,g)$ commutes with every element of $A \rtimes \phi G$ if and only if $b \in A^G$ and $g \in H$. Because $A^G$ commutes with $H$, the center of $A \rtimes \phi G$ is the direct product $A^G \times H$, as desired.